The projections of a convex body

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Abstract
In this paper we present a new proof to the remarcable inequality of Loomis-Whitney see (2),(3). The generaliged inequality is the following.
Let $O.e_1e_2....e_n$ be an orthonormal Coordinate system in E^n and $F_{e_{j_1j_2\cdots j_k}}$ the projections of a convex body F on the k-plane $(e_{j_1}e_{j_2}...e_{j_k})$. By

$$\prod V(F_{e_{j_1j_2\cdot\cdot j_k}})$$

we denote the product of the volumes of the k-projections of the body F on the $\binom{n}{k}$ coordinate planes. We will prove that:

$$V(F) \leq \left[\prod V(F_{e_{j_1j_2\cdot\cdot j_l}})\right]^{\frac{n}{l\binom{n}{l}}} \leq \left[\prod V(F_{e_{j_1j_2\cdot\cdot j_k}})\right]^{\frac{n}{k\binom{n}{l}}}$$

for $i \leq k \leq l \leq n-1$.

Introduction
Let F be a convex body in the Euclidean space E^n and $O.e_1e_2...e_n$ an orthonormal Cartesian system. We denote by v the unit cube and $V_1(F,v) = V(F,F,..F,v)$ the mixed volume of F and v. The cross sectional measure of F in the direction u is given by

$$\sigma_u = nV_1(F,v) = nV(F,F,...F,v)$$

see [1]. It is well known that:

$$V(F,F...F,v) = V(F,F,...F,e_1)+V(F,f,...F,e_2)+....+V(F,F,...F,e_n) = \frac{\sum_1^n \sigma_{e_i}}{n}$$

From the Brunn MInkowski theorem follows:

$$V(F)^{n-1}V(v) = V(F,F,...F)^{n-1}V(v) \leq V(F,F,...F,v)^n$$

so we have:

$$V(F)^{n-1} \leq \left[\frac{\sum_1^n \sigma_{e_i}}{n}\right]^n$$

but from M.A-G.A inequality we have:

$$\prod \sigma_{e_i} \leq \left[\frac{\sum_1^n \sigma_{e_i}}{n}\right]^n$$

so the question is where is the position of the product $\prod_1^n \sigma_{e_i}$ relative to the $V(F)^{n-1}$?

By the present paper we give the answer. We will prove that:

$$V(F)^{n-1} \leq \prod \sigma_{e_i}$$

and in our formulation, the below

theorem 1

$$V(F)^{n-1} \leq \prod V(F_{e_{j_1 j_2 .. j_{n-1}}}).$$

Additionally, from the above we will prove that:

Theorem 2

$$V(F) \leq \left[\prod V(F_{e_{j_1 j_2 .. j_l}})\right]^{\frac{n}{l\binom{n}{l}}} \leq \left[\prod V(F_{e_{j_1 j_2 .. j_k}})\right]^{\frac{n}{k\binom{n}{l}}}$$

for $i \leq k \leq l \leq n-1$ and by $\prod V(F_{e_{j_1 j_2 .. j_p}})$ we denote the product of the volumes of the $\binom{n}{p}$ projections of F on the p coordinate planes $O.e_{j_1} e_{j_2} ... e_{j_p}$.

Proof of the theorem 1.

For the simplicity we lightly change the formulation, so we will prove:

Let F be a convex body in E^n and $(Ox_1 x_2 .... x_n)$,$(L_1 L_2 .... L_n)$ the Cartesian coordinates system and the Cartesian planes respectively. We denote by $F_1, F_2, ...F_n$ the projections of F to the Cartesian planes. We will prove that:

$$\Big[V(F)\Big]^{n-1} \leq V(F_1) \cdot V(F_2) \cdots V(F_n)$$

where $V(K)$ is the volume of the body K.

We consider $p+1$ hyperplanes ($(n-1)$linear spaces) of equal breadth parallel to L_1. The body is partitioned into p slices $K_1, K_2, ...K_p$ of breadth d. We also denote by $A_1, A_2, ...A_p$ the middle intersections of the slices $K_1, K_2, ...K_p$ with planes parallel to L_1. Let

$$V_i' = d \cdot V(A_i), \qquad V' = \sum_{i=1}^{p} V_i'$$

We will have:

$$V(F) = lim_{d\to 0 p\to\infty} V'$$

The A_i are convex body of $n-1$ dimension so following the method of the mathematical induction we can suppose that the theorem is correct (for the $A_i, i = 1, 2, ...p$). Suppose now that $x_{2i}, x_{3i}, ...x_{ni}$ are the volumes of the projections of the A_i on L_i for $i = 2, 3, ...n$.
According the supposition of the induction

$$[V(A_i)]^{n-2} \leq x_{2i}x_{3i}....x_{ni} = S_i \tag{1}$$

for $i = 2, 3, ...n$
Also

$$V(A_i) \leq V_1(F) \tag{2}$$

From Holder's inequality follows:

$$\left[(\Pi_{i=2}^n x_{i1})^{\frac{1}{n-1}} + (\Pi_{i=2}^n x_{i2}^{\frac{1}{n-1}} + ...(\Pi_{i=2}^n x_{ip})^{\frac{1}{n-1}}\right]^{n-1} \leq (\sum_{i=1}^{p} x_{2i})(\sum_{i=1}^{p} x_{3i})....(\sum_{i=1}^{p} x_{ni})$$

or

$$d^{n-1}\left[(\Pi_{i=2}^n V(F_1)x_{i1})^{\frac{1}{n-1}} + (\Pi_{i=2}^n V(F_1)x_{i2}^{\frac{1}{n-1}} + ..(\Pi_{i=2}^n V(F_1)x_{ip})^{\frac{1}{n-1}}\right]^{n-1}$$

$$\leq V(F_1)d^{n-1}(\sum_{i=1}^{p} x_{2i})(\sum_{i=1}^{p} x_{3i})..(\sum_{i=1}^{p} x_{ni})$$

and from (1),(2) arises:

$$\left[dV(A_1) + dV(A_2) + .. + dV(A_p)\right]^{n-1} \leq V(F_1)(\sum_{i=1}^{p} dx_{2i})(\sum_{i=1}^{p} dx_{3i})..(\sum_{i=1}^{p} dx_{ni})$$

Therefore, for $d \to 0, \;\; p \to \infty$ we finally take:

$$\left[V(F)\right]^{n-1} \leq V(F_1) \cdot V(F_2) \cdots V(F_n)$$

Proof of the Theorem 2

First we prove that:

$$V(F) \leq \left[\prod V(F_{e_{j_1 j_2 .. j_k}})\right]^{\frac{n}{k\binom{n}{k}}},$$

for $1 \leq k \leq n-1$.

From the Theorem 1 in E^{k+1} holds:

$$V^k(F_{e_{j_1 j_2 .. j_{k+1}}}) \leq \prod V(F_{e_{j_1 j_2 .. j_k}}) \tag{3}$$

In the E^n the number of the planes $e_{j_1} e_{j_2} ... e_{j_{k+1}}$ are $\binom{n}{k+1}$. In every inequality from the above (1) there are $k+1$ equal terms to $V(F_{e_{j_1 j_2 .. j_k}})$, therefore

$$\frac{(k+1)\binom{n}{k+1}}{\binom{n}{k}} = n-k$$

non equal terms.

So, multiplying the above $\binom{n}{k+1}$ inequalities, we take:

$$\prod V(F_{e_{j_1 j_2 .. j_{k+1}}})^{\frac{k}{n-k}} \leq \prod V(F_{e_{j_1 j_2 .. j_k}}) \tag{4}$$

The above argument also ensures:

$$\prod V(F_{e_{j_1 j_2 .. j_{k+2}}})^{\frac{k+1}{n-k-1}} \leq \prod V(F_{e_{j_1 j_2 .. j_{k+1}}})$$

and an easy induction leads as finally to:

$$\left[V(F)\right]^{\frac{k(k+1)...(n-1)}{1.2.3..(n-k)}} \leq \prod V(F_{e_{j_1 j_2 .. j_k}}),$$

that is:

$$V(F) \leq \left[\prod V(F_{e_{j_1 j_2 .. j_k}})\right]^{\frac{n}{k\binom{n}{k}}},$$

Let now $k \leq l$. From (4) we take:

$$\prod V(F_{e_{j_1 j_2 .. j_{k+1}}})^{\frac{k}{n-k}} \leq \prod V(F_{e_{j_1 j_2 .. j_k}})$$

or

$$\prod V(F_{e_{j_1 j_2 .. j_{k+2}}})^{\frac{k(k+1)}{(n-k)(n-k-1)}} \leq \prod V(F_{e_{j_1 j_2 .. j_{k+1}}})^{\frac{k}{n-k}}$$

We continue this way and we take:

$$\prod V(F_{e_{j_1 j_2 .. j_l}})^q \leq \prod V(F_{e_{j_1 j_2 .. j_k}})$$

where

$$q = \frac{k(k+1)....(l-1)}{(n-k)....[n-(l-1)]} = \frac{\frac{n}{l\binom{n}{l}}}{\frac{n}{k\binom{n}{k}}}$$

Therefore it follows:

$$V(F) \leq \left[\prod V(F_{e_{j_1 j_2 .. j_l}})\right]^{\frac{n}{l\binom{n}{l}}} \leq \left[\prod V(F_{e_{j_1 j_2 .. j_k}})\right]^{\frac{n}{k\binom{n}{l}}}$$

References

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