

Some propositions on Simplices

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We present in this paper some propositions for a n -dimensional simplex.

We use the following symbols.

The points $A_1, A_2, A_3, \dots, A_{n+1}$ are the vertexes of the n -dimensional simplex $\sigma^{(n)}$ in \mathbb{R}^m .

$V^{(n)}$: The n -dimensional linear measure of $\sigma^{(n)}$. (The volume in Euclidean metric space).

$\sigma_i^{(n-1)}$: The $(n-1)$ -dimensional simplex (or the $(n-1)$ -face) opposite to the vertex A_i .

$V_i^{(n-1)}$: The volume of the $\sigma_i^{(n-1)}$.

$a_{ij} = a_{ji}$: The edge $A_i A_j$.

If $p \in \sigma^{(n)}$ $V_i^{(n)}(p)$ is the volume of the simplex with vertexes $A_1, A_2, \dots, A_{i-1}, p, A_{i+1}, \dots, A_{n+1}$.

If $p_i \in \sigma_i^{(n-1)}$ $V_j^{(n-1)}(p_i)$ is the volume of the simplex $A_1, A_2, \dots, A_{i-1}, A_{i+1}, \dots, A_{j-1}, p_i, A_{j+1}, \dots, A_{n+1}$.

h_i : The distance of the vertex A_i from the $\sigma_i^{(n-1)}$, and d_i the distance of the point $p \in \sigma^{(n)}$ from $\sigma_i^{(n-1)}$.

$A_i p = x_i$, $pp_i = u_i$.

Proposition 1.

Let $\sigma^{(n)}$ be a simplex in $\mathbb{R}^{(n)}$ with vertices A_1, A_2, \dots, A_{n+1} . If p is an interior point and $p_i = A_i p \cap \sigma_i^{(n-1)}$, the product

$$V_{j_1}^{(n-1)}(p_1) V_{j_2}^{(n-1)}(p_2) \dots V_{j_{n+1}}^{(n-1)}(p_{n+1})$$

is constant for $j_i \in \{1, 2, 3, \dots, n+1\} \wedge j_e \neq j_k \wedge j_l \neq \lambda$.

PROOF:

We use the barycentric coordinates of the affine geometry.

If $p \in \sigma^{(n)}$ (interior point) we have

$$p = \sum_{i=1}^{n+1} \lambda_i A_i \quad (1)$$

where $\lambda_i > 0$ and $\sum_{i=1}^{n+1} \lambda_i = 1$.

From (1) we obtain

$$p = \lambda_i A_i + (1 - \lambda_i) \sum_{\substack{j=1 \\ j \neq i}}^{n+1} \frac{\lambda_j A_j}{1 - \lambda_i} \quad (2)$$

and consequently

$$p_i = \sum_{\substack{j=1 \\ j \neq i}}^{n+1} \frac{\lambda_j A_j}{1 - \lambda_i} \quad (3)$$

2. Lemma.

If p is an interior point in a $\sigma^{(m)}$ -simplex A_1, A_2, \dots, A_{m+1} and $p = \sum_{i=1}^{m+1} q_i A_i$ ($q_i > 0, \sum_{i=1}^m q_i = 1$) the ratio $\frac{V_i^{(m)}(p)}{q_i}$ has the same arithmetical value for $i \in \{1, 2, \dots, m+1\}$.

In to prove this we use the method of the mathematical induction.

For $m=2$ the proof is elementary.

We assume that the proposition is true for $m=k$ and let $p_i = A_i p \cap \sigma_i$.

We have $p_i = \sum_{\substack{j=1 \\ j \neq i}}^{k+1} \frac{q_j}{1 - q_i} A_j$ and from the assumption for the $\sigma_i^{(k-1)}$ -simplex we have

$$\frac{V_i^{(k-1)}(p_i)}{q_1} = \frac{V_2^{(k-1)}(p_i)}{q_2} = \dots = \frac{V_{i-1}^{(k-1)}(p_i)}{q_{i-1}} = \frac{V_{i+1}^{(k-1)}(p_i)}{q_{i+1}} = \dots = \frac{V_{n+1}^{(k-1)}(p_i)}{q_{n+1}} \quad (4)$$

We multiply successively the relations (4) with $\frac{h_i}{\kappa}$ and $\frac{d_i}{\kappa}$ and subtract the corresponding terms. Taking into account the relation

$$\frac{1}{\kappa} h_i V_e^{(k-1)}(p_i) - \frac{1}{\kappa} d_i V_e^{(k-1)}(p_i) = V_e^\kappa(p) \quad (e \neq i)$$

we have

$$\frac{V_1^{(k)}(p)}{q_1} = \frac{V_2^{(k)}(p)}{q_2} = \dots = \frac{V_{i-1}^{(k)}(p)}{q_{i-1}} = \frac{V_{i+1}^{(k)}(p)}{q_{i+1}} = \dots = \frac{V_{n+1}^{(k)}(p)}{q_{n+1}}$$

We work now in the same way with the simplex $\sigma_\lambda^{(k-1)}$ ($\lambda \neq i$) and we obtain the desired result.

3. We return now to the proof of the theorem. Using the above lemma we have for the simplexes $\sigma_1^{(n-1)}, \sigma_2^{(n-1)}, \dots, \sigma_{n+1}^{(n-1)}$ the relations

$$\begin{aligned} \frac{V_2^{(n-1)}(p_1)}{\lambda_2} &= \frac{V_3^{(n-1)}(p_1)}{\lambda_3} = \frac{V_4^{(n-1)}(p_1)}{\lambda_4} = \dots = \frac{V_{n+1}^{(n-1)}(p_1)}{\lambda_{n+1}} \\ \frac{V_1^{(n-1)}(p_2)}{\lambda_1} &= \frac{V_3^{(n-1)}(p_2)}{\lambda_3} = \dots = \frac{V_{n+1}^{(n-1)}(p_2)}{\lambda_{n+1}} \\ \frac{V_1^{(n-1)}(p_3)}{\lambda_1} &= \dots \end{aligned} \quad (5)$$

We change the order in each line of (5) in such a way that the indices of the λ 's appearing in the denominator to be all different in each column. If now we multiply the terms in each column we obtain.

$$V_{j_1}^{(n-1)}(p_1) V_{j_2}^{(n-1)}(p_2) \dots V_{j_{n+1}}^{(n-1)}(p_{n+1}) \text{ const.}$$

Proposition 2.

Let p be a point in the interior of the simplex $\sigma^{(n)}$ in $\mathbb{R}^{(n)}$ and $p_i = A_i p \cap \sigma_i^{(n-1)}$

We call $V^{(n)}(p)$ the volume of the n -simplex with vertexes p_1, p_2, \dots, p_{n+1} . We will prove that

$$|V^{(n)}(p)| \leq \frac{1}{n^n} |V^{(n)}|.$$

PROOF.

If $\lambda_1, \lambda_2, \dots, \lambda_{n+1}$ are the barycentric coordinates of the point p we have

$$p = \sum_{i=1}^{n+1} \lambda_i A_i \quad (1)$$

and $\lambda_i > 0$, $\sum_{i=1}^{n+1} \lambda_i = 1$.

We know (proposition 1.3)

$$p_i = \sum_{\substack{j=1 \\ j \neq i}}^{n+1} \frac{\lambda_j A_j}{1 - \lambda_i} \quad (2)$$

Let $x_{i1}, x_{i2}, x_{i3}, \dots, x_{in}$ and $a_{i1}, a_{i2}, a_{i3}, \dots, a_{in}$ be the coordinates of the points p_i and A_i in an orthogonal system.

We have from (2)

$$x_{\kappa j} = \sum_{\substack{i=1 \\ i \neq \kappa}}^{n+1} \frac{\lambda_i}{1 - \lambda_\kappa} a_{ij} \quad (3)$$

$\kappa = [1, 2, \dots, n+1]$ and $j = [1, 2, \dots, n]$.

We know that

$$V^{(n)}(p) = \frac{1}{n!} \begin{vmatrix} 1 & x_{11} & x_{12} & \dots & x_{1n} \\ 1 & x_{21} & x_{22} & \dots & x_{2n} \\ 1 & x_{31} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 1 & x_{n+1,1} & \dots & \dots & x_{n+1,n} \end{vmatrix} \quad (4)$$

The substitution of (3) in (4) will give

$$\prod_{i=1}^{n+1} (1-\lambda_i) V^{(n)}(p) = \begin{vmatrix} 0 & \lambda_2 & \lambda_3 & \dots & \lambda_{n+1} \\ \lambda_1 & 0 & \lambda_3 & \dots & \lambda_{n+1} \\ \lambda_1 & \lambda_2 & 0 & \dots & \lambda_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_1 & \lambda_2 & \lambda_3 & \dots & 0 \end{vmatrix} V^{(n)}$$

$$\text{and } \prod_{i=1}^{n+1} (1-\lambda_i) V^{(n)}(p) = (-1)^n n \prod_{i=1}^{n+1} \lambda_i V^{(n)}$$

The relation between the arithmetical and geometrical mean is

$$1 - \lambda_i = \sum_{\substack{j=1 \\ j \neq i}}^{n+1} \lambda_j \geq n \prod_{\substack{j=1 \\ j \neq i}}^{n+1} \lambda_j^{\frac{1}{n}}$$

Therefore

$$|V^{(n)}(p)| \leq \left| \frac{(-1)^n n \prod_{i=1}^{n+1} \lambda_i}{n^{n+1} \prod_{i=1}^{n+1} \lambda_i} \right| |V^{(n)}|$$

and

$$|V^{(n)}(p)| \leq \frac{1}{n^n} |V^{(n)}|.$$

Equality holds if and only if p is the barycenter.

Proposition 3.

Let p be a point in the simplex $\epsilon^{(n)}$. We put $(A_i p) = x_i$, $(p p_i) = u_i$ with $p_i = A_i p \in \epsilon_i^{(n-1)}$. We will prove

$$1. \sum_{i=1}^{n+1} \frac{x_i}{u_i} \geq n(n+1)$$

$$2. \prod_{i=1}^{n+1} x_i \geq n^n \prod_{i=1}^{n+1} u_i$$

$$3. \sum_{i=1}^{n+1} x_i \geq 2 \sum_{j>\lambda}^{1, n+1} u_j^{\frac{1}{2}} u_\lambda^{\frac{1}{2}}$$

$$4. \sum_{i=1}^{n+1} x_i^2 \geq 2 \sum_{i>j}^{1, n+1} u_i u_j + n(n^2-1) \prod_{i=1}^{n+1} u_i^{\frac{n+1}{2}}$$

$$5. \sum_{i>j}^{1, n+1} \frac{x_i x_j}{u_i u_j} \geq \frac{n^2(n+1)}{2}$$

$$6. \sum_{i>j}^{1, n+1} x_i x_j \geq \sum_{i>j}^{1, n+1} u_i u_j + \frac{1}{2} n(n^2-1)(n+1) \prod_{i=1}^{n+1} u_i^{\frac{n+1}{2}}$$

$$7. (n-1) \sum_{i=1}^{n+1} x_i + n \sum_{i=1}^{n+1} u_i \geq 2 \sum_{i>j}^{1, n+1} x_i^{\frac{1}{2}} x_j^{\frac{1}{2}}$$

PROOF.

Let $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_{n+1}$ be the barycentric coordinates of the point p . We have

$$p = \sum_{i=1}^{n+1} \lambda_i A_i \quad \text{and} \quad \lambda_i > 0, \quad \sum_{i=1}^{n+1} \lambda_i = 1.$$

1. Since $p = \lambda_i A_i + (1-\lambda_i) \sum_{j \neq i}^{n+1} \frac{\lambda_j A_j}{1-\lambda_i} = \lambda_i A_i + (1-\lambda_i) p_i$, we have that $\frac{x_i}{u_i} = \frac{1-\lambda_i}{\lambda_i} = \frac{1}{\lambda_i} \sum_{j \neq i}^{n+1} \lambda_j$.

Hence

$$\sum_{i=1}^{n+1} \frac{x_i}{u_i} = \sum_{i>j}^{1, n+1} \left(\frac{\lambda_i}{\lambda_i} + \frac{\lambda_j}{\lambda_i} \right).$$

Taking into account that $\frac{\lambda_i}{\lambda_i} + \frac{\lambda_j}{\lambda_i} \geq 2$ we obtain

$$\sum_{i=1}^{n+1} \frac{x_i}{u_i} \geq \frac{n(n+1)}{2} \cdot 2 = n(n+1)$$

$$2. \frac{\prod_{i=1}^{n+1} x_i}{\prod_{i=1}^{n+1} u_i} = \frac{\prod_{i=1}^{n+1} (1-\lambda_i)}{\prod_{i=1}^{n+1} \lambda_i}, \quad \text{but} \quad 1-\lambda_i = \sum_{j \neq i}^{n+1} \lambda_j \geq n \prod_{j \neq i}^{n+1} \lambda_j^{\frac{1}{n}} \quad \text{and}$$

$$\prod_{i=1}^{n+1} (1-\lambda_i) \geq n^n \prod_{i=1}^{n+1} \lambda_i \quad \text{etc} \dots$$

$$3. \sum_{i>j}^{1, n+1} x_i = \sum_{i>j}^{1, n+1} \left(\frac{\lambda_i}{\lambda_i} u_i + \frac{\lambda_j}{\lambda_j} u_j \right), \quad \text{but} \quad \frac{\lambda_i}{\lambda_i} u_i + \frac{\lambda_j}{\lambda_j} u_j \geq 2 u_i^{\frac{1}{2}} u_j^{\frac{1}{2}};$$

therefore

$$\sum_{i=1}^{n+1} x_i \geq 2 \sum_{\substack{i,j=1 \\ i < j}}^{n+1} u_i^{\frac{1}{2}} u_j^{\frac{1}{2}}$$

$$4. \sum_{i=1}^{n+1} x_i^2 = \sum_{i,j=1}^{n+1} \left(\frac{\lambda_i^2}{\lambda_j^2} u_j^2 + \frac{\lambda_j^2}{\lambda_i^2} u_i^2 \right) + 2 \sum_{\substack{i,j=1 \\ i,j \neq q}}^{n+1} \frac{\lambda_i \lambda_j}{\lambda_q^2} u_q^2, \text{ but}$$

$$\frac{\lambda_i^2}{\lambda_j^2} u_j^2 + \frac{\lambda_j^2}{\lambda_i^2} u_i^2 \geq 2u_i u_j \quad \text{and} \quad 2 \sum_{\substack{i,j=1 \\ i,j \neq q}}^{n+1} \frac{\lambda_i \lambda_j}{\lambda_q^2} u_q^2 \geq n(n-1) \prod_{i=1}^{n+1} u_i^{\frac{n+1}{2}}$$

$$5. \frac{x_i}{u_i} = \frac{1}{\lambda_i} \sum_{\substack{k=1 \\ k \neq i}}^{n+1} \lambda_k \geq \frac{n \prod_{k=1}^{n+1} \lambda_k^{\frac{1}{n}}}{\lambda_i^{\frac{n+1}{n}}}. \quad \text{Therefore} \quad \frac{x_i x_j}{u_i u_j} \geq n^2 \frac{\prod_{k=1}^{n+1} \lambda_k^{\frac{2}{n}}}{\lambda_i^{\frac{n+1}{n}} \lambda_j^{\frac{n+1}{n}}}$$

If we add all the terms for $i, j = 1, 2, \dots, n+1$ for $i > j$ we obtain

$$\sum_{i>j}^{i,j=1} \frac{x_i x_j}{u_i u_j} \geq n^2 \left[\prod_{k=1}^{n+1} \lambda_k^{\frac{2}{n}} \right] \left[\sum_{i>j}^{i,j=1} \lambda_i^{-\frac{n+1}{n}} \lambda_j^{-\frac{n+1}{n}} \right],$$

but

$$\sum_{i>j}^{i,j=1} \lambda_i^{-\frac{n+1}{n}} \lambda_j^{-\frac{n+1}{n}} \geq \frac{n(n+1)}{2} \prod_{k=1}^{n+1} \lambda_k^{-\frac{2}{n}}$$

Therefore

$$\sum_{i>j}^{i,j=1} \frac{x_i x_j}{u_i u_j} \geq \frac{n^3(n+1)}{2} \prod_{k=1}^{n+1} \lambda_k^{\frac{2}{n}} \lambda_k^{-\frac{2}{n}} = \frac{n^3(n+1)}{2}$$

6.

$$x_i x_j = \left[\left(\sum_{\substack{k=1 \\ k \neq i,j}}^{n+1} \lambda_k \right) : (\lambda_i \lambda_j) \right] u_i u_j + u_i u_j \quad \text{because} \quad \sum_{i=1}^{n+1} \lambda_i = 1.$$

$$\text{Consequently} \quad x_i x_j \geq (n-1) \frac{\prod_{k=1}^{n+1} \lambda_k^{\frac{1}{n-1}}}{\lambda_i^{\frac{1}{n-1}} \lambda_j^{\frac{1}{n-1}}} u_i u_j + u_i u_j \quad (1)$$

Using the relation

$$\sum_{i>j}^{i,j=1} \frac{u_i u_j}{\lambda_i^{\frac{1}{n-1}} \lambda_j^{\frac{1}{n-1}}} \geq \frac{n(n+1)}{2} \frac{\prod_{k=1}^{n+1} u_k^{\frac{2}{n-1}}}{\prod_{k=1}^{n+1} \lambda_k^{\frac{2}{(n+1)(n-1)}}} \quad (2)$$

we obtain from (1)

$$\sum_{i>j}^{1, n+1} x_i x_j \geq \sum_{i>j}^{1, n+1} u_i u_j + \frac{n(n-1)}{2} \prod_{k=1}^{n+1} \lambda_k^{-\frac{1}{n+1}} u_k^{\frac{2}{n+1}}, \quad (3)$$

but

$$1 = \sum_{i=1}^{n+1} \lambda_i \geq (n+1) \prod_{k=1}^{n+1} \lambda_k^{\frac{1}{n+1}}$$

and the relation (3) will give

$$\sum_{i>j}^{1, n+1} x_i x_j \geq \sum_{i>j}^{1, n+1} u_i u_j + \frac{n(n+1)(n^2-1)}{2} \prod_{k=1}^{n+1} u_k^{\frac{2}{n+1}}$$

7. We have $u_i = \frac{\lambda_i}{1-\lambda_i} x_i$. Therefore

$$\begin{aligned} (n-1) \sum_{i=1}^{n+1} x_i + n \sum_{i=1}^{n+1} \frac{\lambda_i}{1-\lambda_i} x_i &= \sum_{i=1}^{n+1} \frac{(n-1)(1-\lambda_i) + n\lambda_i}{1-\lambda_i} x_i = \sum_{i=1}^{n+1} \frac{n-(1-\lambda_i)}{1-\lambda_i} x_i = \\ &= n \sum_{i=1}^{n+1} \frac{x_i}{1-\lambda_i} - \sum_{i=1}^{n+1} x_i, \end{aligned}$$

but $n = \sum_{i=1}^{n+1} (1-\lambda_i)$ and we obtain (Schwarz)

$$n \sum_{i=1}^{n+1} \frac{x_i}{1-\lambda_i} \geq \left[\sum_{i=1}^{n+1} x_i^{\frac{1}{2}} \right]^2 = \sum_{k=1}^{n+1} x_k + 2 \sum_{i>j}^{1, n+1} x_i^{\frac{1}{2}} x_j^{\frac{1}{2}},$$

or

$$(n-1) \sum_{i=1}^{n+1} \lambda_i + n \sum_{i=1}^{n+1} u_i \geq 2 \sum_{i>j}^{1, n+1} x_i^{\frac{1}{2}} x_j^{\frac{1}{2}}.$$

Proposition 4.

Let $\epsilon^{(n)}$ be a n -simplex in $\mathbb{R}^{(n)}$ and p a point in the interior of $\epsilon^{(n)}$. Let the midpoints of the segments $A_1 p, A_2 p, \dots, A_{n+1} p$ be $\mu_1, \mu_2, \dots, \mu_{n+1}$. We consider the hyperplane π_i which pass through μ_i and is parallel to hyperplane in which lies the $\epsilon_i^{(n-1)}$. This cuts from $\epsilon^{(n)}$ a n -simplex which has A_i as one of its vertexes. If $V_{\mu_i}^{(n)}$ is the volume of this

simplex, we shall prove

$$\sum_{i=1}^{n+1} V_{\mu_i}^{(n)} \geq \left(\frac{n}{2}\right)^2 \frac{V^{(n)}}{(n+1)^{n-1}}.$$

PROOF.

We set $p = \sum_{i=1}^{n+1} \lambda_i A_i$ with $\lambda_i > 0$ and $\sum_{i=1}^{n+1} \lambda_i = 1$. We have

$$\mu_i = \frac{1}{2}(A_i + p) = \frac{1}{2} [\lambda_1 A_1 + \lambda_2 A_2 + \dots + \lambda_{i-1} A_{i-1} + (1 + \lambda_i) A_i + \lambda_{i+1} A_{i+1} + \dots + \lambda_{n+1} A_{n+1}]$$

and

$$\frac{(A_i \mu_i)}{(A_i p_i)} = \frac{1 - \lambda_i}{2} \quad (p_i = A_i p \cap \sigma_i^{(n-1)}).$$

Therefore

$$V_{\mu_i}^{(n)} = \left[\frac{1 - \lambda_i}{2}\right]^n V^{(n)}.$$

We sum with respect to i from 1 to $n+1$ and we obtain

$$\sum_{i=1}^{n+1} V_{\mu_i}^{(n)} = \frac{1}{2^n} V^{(n)} \sum_{i=1}^{n+1} (1 - \lambda_i)^n$$

Taking into account the relation

$$\sum_{i=1}^{n+1} (1 - \lambda_i)^n \geq \frac{\left[\sum_{i=1}^{n+1} (1 - \lambda_i)\right]^n}{(n+1)^{n-1}} = \frac{n^n}{(n+1)^{n-1}}$$

we obtain the desired result.

Equality holds if and only if p is the bary-center.

Proposition 5.

Let $\sigma^{(n)}$ be a n -simplex in R^n with vertexes A_1, A_2, \dots, A_{n+1} and $\sigma^{(q)}$ a q -face with vertexes A_1, A_2, \dots, A_{q+1} ($q < n$). If $p = \sum_{i=1}^{n+1} \lambda_i A_i$, $\lambda_i \geq 0$, $\sum_{i=1}^{n+1} \lambda_i = 1$, and

$q = \sum_{j=1}^{q+1} \mu_j A_j$ we shall prove that

$$|p - q| \leq \sum_{i=1}^{n+1} \sum_{j=1}^{q+1} \lambda_i \mu_j d$$

with $i \neq j$, where d is the diameter of $\mathcal{S}^{(n)}$.

PROOF.

$$\text{We have } q - p = \sum_{j=1}^{q+1} \mu_j A_j - p = \sum_{j=1}^{q+1} \mu_j A_j - \sum_{j=1}^{q+1} \mu_j p = \sum_{j=1}^{q+1} \mu_j (A_j - p). \quad (1)$$

Using

$$A_j - p = \sum_{i=1}^{n+1} \lambda_i A_j - \sum_{i=1}^{n+1} \lambda_i A_i = \sum_{i=1}^{n+1} \lambda_i (A_j - A_i)$$

we obtain from (1)

$$q - p = \sum_{j=1}^{q+1} \sum_{i=1}^{n+1} \lambda_i \mu_j (A_j - A_i) \quad (2)$$

The number of the terms in the right-hand side of (2) is $(q+1)(n+1)$. Of this summand the $q+1$ are equal to zero, and $|A_j - A_i| \leq d$. So we have

$$|q - p| \leq \sum_{i=1}^{n+1} \sum_{j=1}^{q+1} \lambda_i \mu_j d$$

for $i \neq j$. If we take for p, q the centroids, then the above relation reduces to $|q - p| \leq \frac{n}{n+1} d$.

(ALEKSANDROV: Combinatorial Topology page 214)

The Leibnitz formula and $\mathcal{S}^{(n)}$ -simplex.

Let A_1, A_2, \dots, A_n be n points in $R^{(d)}$ with masses m_1, m_2, \dots, m_n and let G be the barycenter. (That is for a point o $\vec{oG} = \frac{m_1 \vec{oA}_1 + m_2 \vec{oA}_2 + \dots + m_n \vec{oA}_n}{m_1 + m_2 + \dots + m_n}$)

If p is any point in $R^{(d)}$, the following relation holds

$$\sum_{i=1}^n m_i \vec{pA}_i^2 = m \vec{pG}^2 + \sum_{i=1}^n m_i \vec{GA}_i^2 \quad (\text{Leibnitz formula}) \quad (1)$$

$$m = m_1 + m_2 + \dots + m_n.$$

We set $(A_i A_j) = (A_j A_i) = a_{ij} = a_{ji}$.

We shall prove the following formula

$$\sum_{i=1}^n m_i \overrightarrow{pA_i}^2 = m \overrightarrow{pG}^2 + \frac{1}{m} \sum_{i,j}^{1,2} m_i m_j a_{ij}^2. \quad (2)$$

It suffices to prove that

$$\sum_{i=1}^n m_i \overrightarrow{GA_i}^2 = \frac{1}{m} \sum_{i,j}^{1,2} m_i m_j a_{ij}^2. \quad (3)$$

We use the mathematical induction.

The relation (3) is elementary for $n=2$. So for any two points $A_1(m_1)$ and $A_2(m_2)$ we have

$$m_1 \overrightarrow{GA_1}^2 + m_2 \overrightarrow{GA_2}^2 = \frac{m_1 m_2 a_{12}^2}{m_1 + m_2}. \quad (4)$$

We suppose that the proposition holds for $n=k$.

Let G be the barycenter of the points $A_1(m_1), A_2(m_2), \dots, A_k(m_k)$ and G' the barycenter of the points $A_1(m_1), A_2(m_2), \dots, A_k(m_k), A_{k+1}(m_{k+1})$.

For the two points G ($m = m_1 + m_2 + \dots + m_k$) and $A_{k+1}(m_{k+1})$ we have, according to (4)

$$m \overrightarrow{G'G}^2 + m_{k+1} \overrightarrow{G'A_{k+1}}^2 = \frac{m m_{k+1} \overrightarrow{GA_{k+1}}^2}{m + m_{k+1}}. \quad (5)$$

We put in (2) $p = G'$ and obtain

$$\sum_{i=1}^k m_i \overrightarrow{G'A_i}^2 = m \overrightarrow{G'G}^2 + \frac{1}{m} \sum_{i,j}^{1,k} m_i m_j a_{ij}^2. \quad (6)$$

We add (5) and (6)

$$\sum_{i=1}^{k+1} m_i \overrightarrow{G'A_i}^2 = \frac{m m_{k+1} \overrightarrow{GA_{k+1}}^2}{m + m_{k+1}} + \frac{1}{m} \sum_{i,j}^{1,k} m_i m_j a_{ij}^2. \quad (7)$$

We put in relation (2) $p = A_{k+1}$ and we solve for $\overrightarrow{GA_{k+1}}^2$ and we substitute it in (7). After the simplifications we have

$$\sum_{i=1}^{n+1} m_i \vec{GA}_i^2 = \frac{1}{m+m_{n+1}} \sum_{\substack{i,j \\ i < j}}^{i,j=1} m_i m_j a_{ij}^2$$

We now apply the formulas (1) and (2) to a $\sigma^{(n)}$ -simplex.

We consider the $\sigma^{(n)}$ -simplex in $R^{(n)}$ with vertices A_1, A_2, \dots, A_{n+1} and let p be an interior point.

We have $p = \sum_{i=1}^{n+1} \lambda_i A_i$ and $\lambda_i > 0$, $\sum_{i=1}^{n+1} \lambda_i = 1$.

The formula (1) for a point q becomes

$$\sum_{i=1}^{n+1} \lambda_i \vec{qA}_i^2 = \vec{pq}^2 + \sum_{i=1}^{n+1} \lambda_i \vec{pA}_i^2 \quad (8)$$

or

$$\sum_{i=1}^{n+1} \lambda_i \vec{qA}_i^2 \geq \sum_{i=1}^{n+1} \lambda_i \vec{pA}_i^2 \quad (9)$$

Equality for $q = p$.

If q coincides with the center of the n -circum-sphere and $(qA_i) = R$ we have

$$R^2 \geq \sum_{i=1}^{n+1} \lambda_i \vec{pA}_i^2 \quad (10)$$

If we use the formula (2), we obtain

$$\sum_{i=1}^{n+1} \lambda_i \vec{qA}_i^2 = \vec{pq}^2 + \sum_{\substack{i,j \\ i < j}}^{i,j=1} \lambda_i \lambda_j a_{ij}^2 \quad (11)$$

$$\sum_{i=1}^{n+1} \lambda_i \vec{qA}_i^2 \geq \sum_{\substack{i,j \\ i < j}}^{i,j=1} \lambda_i \lambda_j a_{ij}^2 \quad (12)$$

$$R^2 \geq \sum_{\substack{i,j \\ i < j}}^{i,j=1} \lambda_i \lambda_j a_{ij}^2 \quad (13)$$

A large number of inequalities can be obtained from the relations 9, 10, 11, 12 and 13 e.g.

$$a) \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} \lambda_j a_{ij}^2 > (n+1) \sum_{i=1}^{n+1} \lambda_i p A_i^2$$

$$b) \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} \lambda_i a_{ij}^2 > (n+1) \sum_{i>j}^{1, n+1} \lambda_i \lambda_j a_{ij}^2$$

$$c) R \gg \frac{\sum_{i>j}^{1, n+1} a_{ij}}{(n+1)\sqrt{n(n+1)}}$$

$$d) R \gg \frac{1}{S} \left[\sum_{i>j}^{1, n+1} V_i^{(n-1)} V_j^{(n-1)} a_{ij}^2 \right]^{\frac{1}{2}} \quad \text{with} \quad S = \sum_{i=1}^{n+1} V_i^{(n-1)} \quad \text{etc.}$$

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