

John's matrix. The vector relative to simplex.

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John's matrix is a very interesting and fruitful subject. The idea of the vector relative to simplex seems to simplify some theorems and propositions. The following includes some of my results without the proofs. Let $s = A_1A_2\dots A_{n+1}$ be a simplex in E^n . It is well known that there exists real positive numbers c_1, c_2, \dots, c_{n+1} so that:

$$\sum_{i=1}^{n+1} c_i u_i = 0, \quad \sum_{i=1}^{n+1} c_i = n,$$

where u_i the unit vectors perpendiculars to the facets of s and directed to the exterior of s .

We suppose that a is a vector in E^n . We define the vector Da relative to simplex s as follows:

$$Da = \sum_{i=1}^{n+1} c_i (u_i, a) u_i \quad (1)$$

The following properties can be easily proved.

1. $\forall \lambda \in \mathfrak{R} : \lambda Da = D(\lambda a)$
2. $D(a + b) = Da + Db$
3. $a = 0 \Leftrightarrow Da = 0$
4. $(Da, b) = (a, Db) = (Db, a)$
5. $(Da, b_1) + (Da, b_2) = (Da, b_1 + b_2)$
6. $(Da, Db) = \sum_{i,j}^{1,n+1} c_i c_j (u_i, a) (u_j, b) (u_i, u_j)$

John's Matrix

Let $O.x_1x_2\dots x_n$ be the Cartesian orthogonal system with unit vectors e_1, e_2, \dots, e_n . The operator $A = \sum_{i=1}^{n+1} c_i u_i \otimes u_i$ where :

Some Propositions

1. John's theorem for the simplex.

The Euclidean ball is the ellipsoid of maximum volume contained in the simplex $s = A_1A_2\dots A_{n+1}$ if and only if:

(a). $\sum_1^{n+1} c_i u_i = 0$

(b). $\sum_1^{n+1} c_i u_i \otimes u_i = I_n$ the identity matrix.

The u_i and c_i as already defined.

2. For the inscribed sphere (I, r) in the simplex s holds:

$$r \leq \frac{1}{\lambda_n}$$

3. Let

$$s_1 = \sum_1^n \frac{x_i^2}{a_i^2} - 1$$

be the inscribed ellipsoid in s , then

$$a_1 a_2 \dots a_n \leq \frac{1}{\lambda_n^n}$$

4. For the circumscribed sphere (O, R) of s holds:

$$R \leq \frac{n}{\lambda_n}.$$

Generally, it holds:

$$nr \leq R \leq \frac{n}{\lambda_n}.$$

5.

$$\lambda_1 \geq 1 \geq \lambda_n$$

6. Let $\sigma = A_1A_2\dots A_{n+1} \cap p$ the intersection of the simplex s with the $(n - 1)$ plane p and (K, r) the inscribed sphere B_{n-1}^2 in σ . We proved that:

$$r^2 \leq \frac{n(\lambda_n^2 + n\lambda_1)}{(n - \lambda_1)(\lambda_n^2 + n\lambda_1 - \lambda_1^2)}$$

References

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