A characteristic property of the ellipse

G.Tsintsifas

1. Introduction

We suppose that (c) is an ellipse and A, B the foci. We consider the tangent lines from an exterior point M to (c), MM_1 and MM_2. It is well known,see [1], that the angles AMM_1 and BMM_2 are equal. From the above arises the question: Is this property a characterization of the ellipse?
More clear the problem is:
Let (k) be a rotund closed convex curve and A, B interior points. We consider, from an external point M the two support lines MM_1, MM_2 and we assume that for every point M, the angles AMM_1 and BMM_2 are equal. In this note we answer to the question whether (k) is an ellipse. The answer is positive, so we will prove that the convex curve (k) must be an ellipse and the above property is a characterization of the ellipse.

2. Proof

Let (ϵ) be a support line of (k). We easily find that the product of the distances from A, B to (ϵ) is constant.
Indeed. Assume (η) the support line parallel to AB and we denote: $M = (\eta) \cap (\epsilon)$.

We drow the perpendiculars AA_1, BB_1 to (ϵ) and AA', BB' to (η). We will have:

$$angle AMA_1 = angle BMB' \;\; and \;\; angle AMA' = angle BMB_1,$$

therefore the triangles AMA_1, BMB' are similar as well as the triangles AMA', BMB_1. Hence,

$$\frac{BB'}{AA_1} = \frac{BM}{AM} \;\; and \;\; \frac{AA'}{BB_1} = \frac{AM}{BM},$$

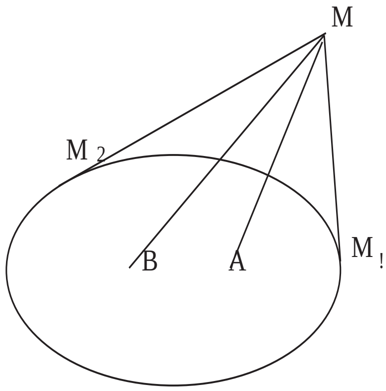

Figure 1:

that is:

$$AA_1.BB_1 = BB'.AA' = b^2 \tag{1}$$

where $b = AA' = BB'$.

We now consider the Kartesian orthogonal system with origin the middle point O of AB and Ox axis the line AB. Let $p(\theta)$ the support function of (k) relative to the point O and θ the angle of the normal of the point O from a line (ϵ). The equation of (ϵ) is:

$$x\cos\theta + y\sin\theta = p(\theta), \tag{2}$$

where $p(\theta)$ is the distance of the point O from (ϵ).
But, then we have:

$$AA_1 = |c\cos\theta - p(\theta)|, \quad BB_1 = |-c\cos\theta - p(\theta)|$$

where $A(c,0), B(-c,0)$. So, from the above we have:

$$AA_1.BB_1 = b^2 = p^2(\theta) - c^2\cos^2(\theta) \tag{3}$$

Let $P(x,y) = (\epsilon) \cap (k)$. It is well known that:

$$x = p(\theta)\cos(\theta) - \acute{p}(\theta)\sin\theta \qquad y = p(\theta)\sin\theta) + \acute{p}(\theta)\cos\theta \tag{4}$$

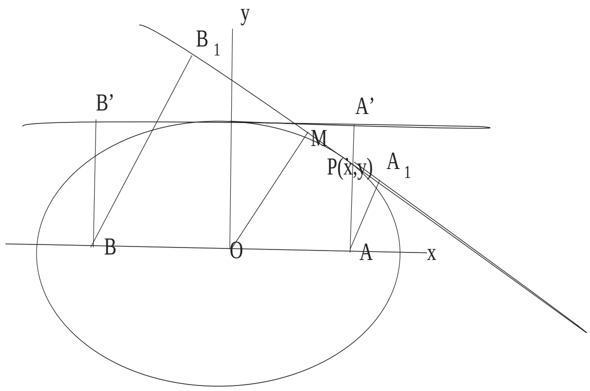

Figure 2:

see [2].
Puting $c^2 = a^2 - b^2$, from (3) we have:

$$p^2(\theta) = a^2 \cos^2 \theta + b^2 \sin^2 \theta. \tag{5}$$

From (4) and (5) we take:

$$x = \frac{a^2 \cos \theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}}$$

$$y = \frac{b^2 \sin \theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}}$$

and finally:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

the equation of the curve (k), that is the equation of an ellipse.

References.

1. C. Smith, Conic Sections Art.230, MacMillan and Co LTD, London 1956.
2. F. Valentine p. 160, McGraw-Hill book Company, 1964.