Lattice point covering property of the triangle.

G. A. Tsintsifas Theassaloniki Greece GR

1. Introduction

Ivan Niven and H.S.Zackkerman in [1] discussed the problem of the lattice point covering of the triangle. In this note we are working in the same problem, following another formulation and we obtain a remarkable Geometric solution.

2. Theorem.

Let T be a triangle in the lattice plane and suppose that the minimum inscribed square has a side bigger or equal to 1. Then T has the lattice point covering property.

Proof

It is easy to see that there are at most three squares inscribed in the triangle T, each of every side. An elementary calculation shows that the smaller inscribed square corresponds to the biggest side.

Indeed. The square $EFZH$ corresponds to the side $BC = a$. From the similar triangles FAZ, CAB, and BZH, BAD we have:
$\frac{AI}{AD} = \frac{ZF}{BC}$ or $\frac{h_a - x}{h_a} = \frac{x}{a}$, that is:

$$x = \frac{ah_a}{a + h_a} \tag{1}$$

Where x is the side of the square and h_a the altitude from A.
We assume now $a \geq b \geq c$. We easily find

$$a + h_a \geq b + h_b \geq c + h_c \tag{2}$$

because of

$$a + \frac{2F}{a} \geq b + \frac{2F}{b}$$

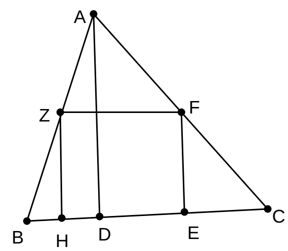

Figure 1:

where F denotes the area of the triangle ABC, or,

$$(a-b)(ab-2F)\geq 0.$$

So, (2) holds.
From (1),(2) we understand that the smaller square corresponds to the biggest side.
Let now $T=ABC$ be a triangle in the lattice plane (Ov,Ou). We consider the parallel lines x_a,x_b,x_c through A,B,C to Ov. One from these must intersect the opposite side. We can suppose, without any restriction, that x_a intersects BC. We consider the inscribed square on BC. This square (L) has sides bigger or equal to 1. That is, from (1)

$$\frac{ah_a}{a+h_a}\geq 1$$

Let $P=x_a\cap BC$. Obviously $h_a\geq 1$, therefore we can find points $M\in AC$, such that the parallel straight segments to AP,MN and SQ have length equal to 1. Let v_1,v_2 the distances of B,C to AP and d_1,d_2 the breadths of the strips MN,AP and SQ,AP. From the similar triangles

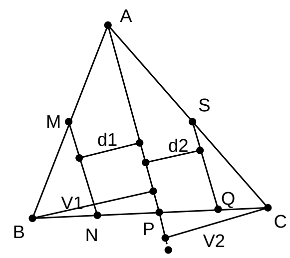

Figure 2:

MBN, PBA we have:

$$\frac{MN}{AP} = \frac{v_1 - d_1}{v_1} \quad or \quad \frac{1}{p} = \frac{v_1 - d_1}{v_1} \quad or \quad d_1 = v_1 - \frac{v_1}{p}$$

where $AP = p$. Similarly we have

$$d_2 = v_2 - \frac{v_2}{p}$$

or,

$$d_1 + d_2 = v_1 + v_2 - \frac{v_1 + v_2}{p} \tag{3}$$

We also have.

$$(v_1 + v_2)p = 2F \tag{4}$$

where F denotes the area of ABC triangle. From (3),(4) we have:

$$d_1 + d_2 = \frac{(p-1)}{p}\frac{2F}{p} = \frac{(p-1)}{p}\frac{ah_a}{p} \geq \frac{p-1}{p}a \tag{5}$$

because $p \geq h_a$. From (5) we have $a^2p + aph_a \geq a^2h_a + aph_a$
or

$$\frac{ap}{a+p} \geq \frac{ah_a}{a+h_a} \geq 1. \tag{6}$$

From the last relation (6) follows

$$\frac{p-1}{p}a \geq 1$$

that is $d_1 + d_2 \geq 1$.
So the parallelogramme of side $MN = SQ = 1$, has the breadth of the strip MN, SQ which is at least 1. Therefore there is a lattice line parallel to Ov intersecting the parallelogramme $MNQS$ in a straight line segment of a length at least 1. So there is a lattice point belonging to the interior of the above parallelogramme.

Reference.
1. Ivan Niven and H.S. Zuckerman, Lattice point covering by plane figures, Amer. Math. Monthly, 74, 1967, p. 353-362. 2. The Geometry of Nambers, C.D.Olds, Anneli Lax, Giuliana Davidof, The Mathematical Association of America.