The maximal inscribed and the minimal circumscribed ellipse for a centrally symmetric convex figure.

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Introduction

An interesting problem in the theory of convex sets is the approximation of the convex sets via regular, with well known properties, figures. John's theorem asserts us that in every centrosymmetric figure F there is an inscribed ellipse E and a circumscribed ellipse $\sqrt{2}E$ including F. In this note we study the maximal inscribed and the minimal circumscribed ellipse of a centrally symmetrc convex figure F and we prove two theorems between the area and the remarkable elements of the figures.

1.The maximal inscribed ellipse

Let AA' be a diameter of the centrosymmetric figure F and O its center. We consider an orthogonal Cartesian coordinates system with origin the point O. The Ox axis coincides with OA and we put:

$$A(p,0), A'(-p,0), B(0,q), B'(0,-q),$$

where $\frac{q}{p} \leq 1$ and B, B' the points of intersection of the perimeter of F with the vertical yOy' axis.
The smaller centrosymmetric convex figure through the points A,A',B,B' is the rhombus ABA'B', hence the smaller inscribed ellipse E_i in F, is the inscribed ellipse E_1 in the rhombus ABA'B'. We consider the affine transformation f, transforming the rhombus ABA'B' to the square $AB_1A'B'_1$ and the

ellipse E_1 to the circle E'_1. The affinity preserves the ratio of areas, therefore we will have:

$$\frac{[E_1]}{[ABA'B']} = \frac{[E'_1]}{[AB_1A'B'_1]} = \frac{\pi p^2}{4p^2} = \frac{\pi}{4} \tag{1}$$

The area of the figure Q is denoted by [Q].

$$\frac{[E_1]}{[ABA'B']} = \frac{[E_1]}{2pq} \tag{2}$$

From (1),(2) follows that

$$E_1 = \frac{\pi pq}{2}$$

The maximum centrosymmetric figure through the points A,A',B,B' is the orthogonal parallelogramme T with the sides on the perpendiculars at the points A,A' to the axes Oy and at the points B,B' to the Ox axes. The inscribed ellipse in T has area πpq, we conclude

Theorem 1

The maximum inscribed ellipse E_i in a centrally symmetric convex figure F completes the inequality.

$$\frac{\pi pq}{2} \leq [E_i] \leq \pi pq$$

The minimal circumscribed ellipse.

As in the first part, we consider a Cartesian system with origin the center O of the centrally symmetric figure F.

Let it be AA' a diameter and we put:

$$A(p, 0), A'(-p, 0).$$

Let now, ϵ, ϵ' the parallel to Ox axis support lines of F and B, B' their intersections with the Oy axis. We put $B(0, q)$ and $B'(0, -q)$. We also suppose that our choice is so that the ratio p/q is minimum.

(a). We assume that $\frac{q}{p} < \frac{\sqrt{2}}{2}$.

The biggest centrally symmetric figure F is included in the convex region $S = AKB'LA'MBN$ bounded by the lines ϵ, ϵ' and two circular arcs

NK, LM, having the point O as a center and radius p.
The point N has coordinates $(\sqrt{p^2 - q^2}, q)$ and the equation of E_2 is:

$$E_2: \ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The minimum ellipse E_2 contains the points N, K, L, M, so we have:

$$\frac{p^2 - q^2}{a^2} + \frac{a^2}{b^2} = 1$$

or,

$$b^2 = \frac{q^2 a^2}{a^2 - (p^2 - q^2}. \tag{3}$$

The area of E_2 is: $[E]^2 = \pi^2 a^2 b^2$.
Hence, using (3) we take:

$$[E]^2 = \pi^2 \frac{q^2 a^4}{a^2 - (p^2 - q^2}. \tag{4}$$

From the formula (5) we easily see that $[E_2]^2$ is continous with respect a^2, $(a^2 > p^2 - q^2)$.
We find the minimum of $[E_2]$ using elementary theorems of calculus. That is from (5) we have:

$$\pi^2 q^2 a^4 - [E_2]^2 a^2 + [E_2]^2 (p^2 - q^2) = 0,$$

Thus, setting

$$\Delta = [E_2]^4 - 4\pi^2 q^2 (p^2 - q^2) \geq 0,$$

we have:

$$[E_2] \geq 2\pi q \sqrt{p^2 - q^2}. \tag{5}$$

The above minimum is taken for

$$a = \sqrt{2(p^2 - q^2)}.$$

Assuming now that, $a > p$ or equivalently $\frac{q}{p} < \frac{\sqrt{2}}{2}$, we conclude that there is an ellipse E_2 with area at most $2\pi q \sqrt{p^2 - q^2}$ including F .

For the case $\frac{p}{q} \geq \frac{\sqrt{2}}{2}$, we see that: $a \leq p$. That is the ellipse E_2 does not contain S. Therefore taking in mind the continuity of $[E_2]$, see (5), we conclude that every ellipse including S has at least area than the crcle (O, p). Hence we end at the following

Theorem 2

The minimum ellipse E_s circumscribed about the centally symmetric convex figure F has area:

1. for $\frac{q}{p} \leq \frac{\sqrt{2}}{2}$

$$[E_s] \geq 2\pi q \sqrt{p^2 - q^2}$$

2. for $\frac{q}{p} \geq \frac{\sqrt{2}}{2}$

$$[E_s] \geq \pi p^2$$

References

1. C. Smith, Conic Sections, Mc Millan, 1965.
2. I. M. Yaglom and V. G. Boltianskii, Convex Figures, Holt Rinechart and Winston, 1961.