Lattice point covering property of the ellipse.

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Ivan Niven and H.S.Zukkerman in [1] proved that the ellipse with axes $2a$, $2b$ has the lattice point covering property if and only if $4a^2b^2 \geq a^2 + b^2$. In this note we will give another formulation and a remarkable simple proof of the above theorem.

1. Lemma.

Let (c) be an ellipse with axes $2a$, $2b$ and center K. A right angle EKF is moving around K, $E \in (c)$ and $F \in (c)$.Let M the foot of the center K on EF. The locus of the point M is the inscribed circle of the inscribed square in the ellipse (c),see[2].

Proof.

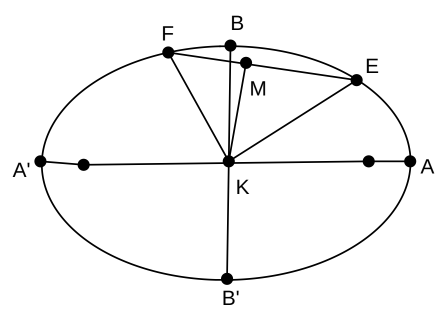

Figure 1:

Let
$$\frac{1}{KE^2}=\frac{1}{r_1^2}=\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}$$
and
$$\frac{1}{KF^2}=\frac{1}{r_2^2}=\frac{\cos^2(\theta+\pi/2)}{a^2}+\frac{\sin^2(\theta+\pi/2)}{b^2}=\frac{\sin^2\theta}{a^2}+\frac{cos^2\theta}{b^2}$$
see [2] page 147.
Therefore
$$\frac{1}{r_1^2}+\frac{1}{r_2^2}=\frac{1}{a^2}+\frac{1}{b^2}=\frac{a^2+b^2}{a^2b^2}$$
But we know from the elementary Geometry
$$\frac{1}{r_1^2}+\frac{1}{r_2^2}=\frac{1}{KM^2}$$
so, we have:
$$KM=\frac{ab}{\sqrt{a^2+b^2}}.$$
It is very simple problem to calculate the side of the inscribed square in (c). The side of the square is
$$\frac{2ab}{\sqrt{a^2+b^2}}$$
so the iscribed circle (q) in the above square has as radius
$$\frac{ab}{\sqrt{a^2+b^2}}.$$
It is elementary to prove the contrary, that is if $E'F'$ is a chord of (c) touching on (q) then the angle $E'KF'$ is a right angle.
We also can prove that
$$EF\geq\frac{2ab}{\sqrt{a^2+b^2}}.$$
We know that:
$$(r_1^2+r_2^2)(\frac{1}{r_1^2}+\frac{1}{r_2^2})\geq 4$$

or,

$$(r_1^2 + r_2^2)\frac{a^2 + b^2}{a^2 b^2} \geq 4$$

or,

$$r_1^2 + r_2^2 = EF^2 \geq \frac{4a^2 b^2}{a^2 + b^2}$$

or,

$$EF \geq \frac{2ab}{\sqrt{a^2 + b^2}}.$$

2. Theorem
The ellipse (c) has the lattice point property if and only if includes a square of side 1.
Proof.
The inscribed square in (c) is the max. included square and has as a side

$$\frac{2ab}{\sqrt{a^2 + b^2}}$$

so we have:

$$\frac{2ab}{\sqrt{a^2 + b^2}} \geq .1$$

Let (q) be the inscribed circle in the inscribed square and Ov, Ou the axes of the lattice plane. We consider $\eta\eta'$, $\omega\omega'$ the parallel to Ov tangent lines to (q). The breadh of the strip $\eta\eta'$, $\omega\omega'$ is at least 1. So, there is a parallel line, to Ov of the lattice plane, SS', intersecting the ellipse (c) at the points T,P interior of the strip. According our lemma $T'P' \geq 1$, see fig. 2, therefore $TP \geq T'P' \geq 1$ and so there is a lattice point on the straight line segment TP.
Also, it holds the contrary, that is, unter the assumption

$$\frac{2ab}{\sqrt{a^2 + b^2}} < 1$$

the ellipse with axes $2a$,$2b$ does not have the lattice point covering property.That is very simple. We just put the max. inscribed square of the ellipse inside and with parallel sides in a lattice square.

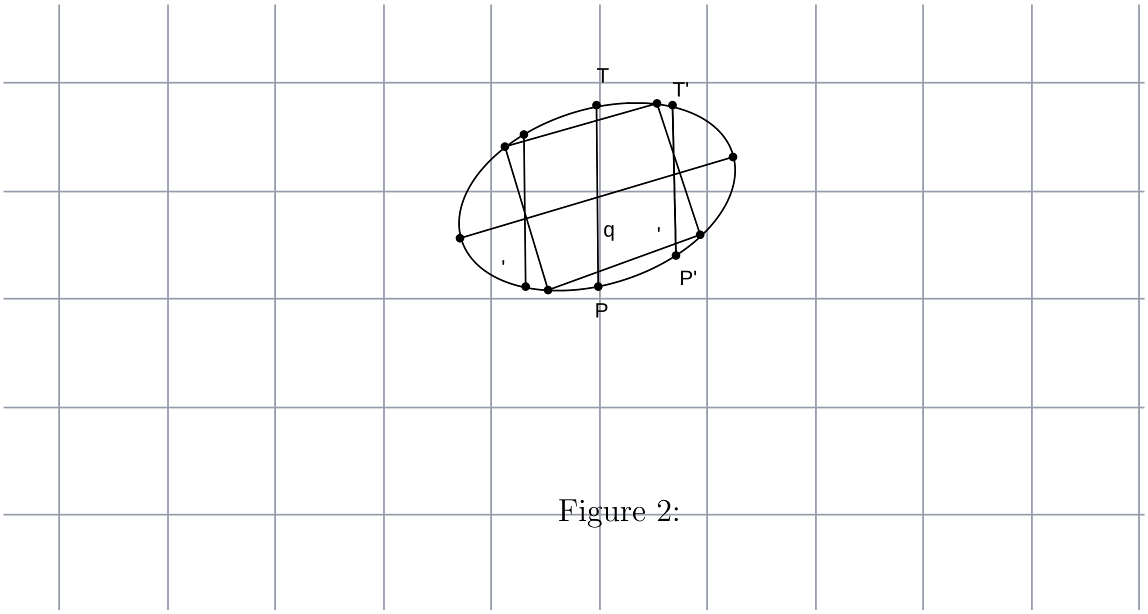

Figure 2:

References.
1. Ivan Niven and H.S. Zuckerman, Lattice point covering by plane figures. Amer. Math. Monthly, 74, 1967, p. 353-362.
2. C. Smith, Conic sections, Macmillan 1956.
3. The Geometry of Nambers, C.D.Olds, Anneli Lax, Giuliana Davidof, The Mathematical Association of America.