The Orthoptic Curve of a convex figure

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1. Introduction

Let F be a convex figure in E^2 and $K = \vartheta F$ the boundary curve of F. We suppose that K is regular convex curve with positive curvature (rotund) and two points P_1, P_2 of K. The support lines at the points P_1, P_2 are intersecting at the point P. Continuity arguments assure that the point set $\{P/\angle P_1 P P_2 = \pi/2$ is a curve denoted byK_*.

Definition We will call K_*, the Orthoptic curve of the curve$K$.

If K is the circle (O, R) then K_* will be the circle $(O, R\sqrt{2})$. If K is the ellipse $x^2/a^2 + y^2/b^2/2 = 1$ then K_* will be concentric circle of radius $\sqrt{a^2 + b^2}$. Here, the main problem for us is to determine the convex curve K supposing that K_* is circle, see also [1]. We will also prove the following interesting inequalities.

$$(\mathbf{a}) \qquad A(K_*) \geq 2A(K)$$

$$(\mathbf{b}) \qquad L(K_*) \geq 2L(K).$$

Where by A we denote the area enclosed by the curves K, K_* and L the length.

2. The problem

We suppose that K_* is a circle of radius 1. We will try to determine the curve K.

Let e_1 the support line at the point $P \in K$ and $p_1 = p(\theta)$ the support function of K for the e_1. It is well known that:

$$\forall (x, y) \in e_1 : \quad x \cos\theta + y \sin\theta = p(\theta) = p_1 \tag{1}$$

see [1].

Let now e_2 the support line, perpendicular to e_1, so that:

$$\forall (x, y) \in e_2 : \quad x \cos(\theta + \pi/2) + y \sin(\theta + \pi/2) = p(\theta + \pi/2) = p_2 \tag{2}$$

From the system of (1) and (2) we can find the coordinates of the point $M = e_1 \cap e_2$. We have:

$$x = x_M = p_1 \sin\theta + p_2 \cos\theta, \quad y = y_M = p_1 \cos\theta + p_2 \sin\theta \tag{3}$$

Easily from (3) we find:

$$|OM|^2 = p_1^2 + p_2^2$$

or,

$$p^2(\theta) + p^2(\theta + \pi/2) = 1. \tag{4}$$

The function $p^2(\theta)$ is periodic with period 2π. Assuming that the conditions for the expansion in Fourier series exist, we will have:

$$p^2(\theta) = \frac{1}{2}a_0 + \sum_1^\infty (a_n \cos n\theta + b_n \sin n\theta)$$

$$p^2(\theta + \pi/2) = \frac{1}{2}a_0 + \sum_1^\infty (d_n \cos n\theta + g_n \sin n\theta)$$

where,

$$d_n = a_n \cos\frac{n\pi}{2} + b_n \sin\frac{n\pi}{2}$$

$$g_n = b_n \cos\frac{n\pi}{2} - a_n \sin\frac{n\pi}{2}.$$

For, $a_0 = 1, \ \ a_n + d_n = 0, \ \ b_n + g_n = 0, \quad n \geq 1$, we find:
$a_1 = b_1 = 0, \ \ a_3 = b_3 = 0, \ \ a_4 = b_4 = 0, \ \ a_5 = b_5 = 0, \ \ a_7 = b_7 = 0$ ...,
that is $a_{4k+i} = b_{4k+i} = 0$ for $i = 0, 1, 3,$ k natural number. We can arbitrarely define $a_{4k+2}, \ b_{4k+2}$. Therefore if the support function is:

$$p^2(\theta) = \frac{1}{2} + \sum_1^\infty \left[a_{4k+2}\cos(4k+2)\theta + b_{4k+2}\sin(4k+2)\theta\right]$$

then we will have:

$$p^2(\theta) + p^2(\theta + \pi/2) = 1.$$

An example.
We can take the curve (c) with support function

$$p(\theta) = \sqrt{\frac{1}{2} + a\cos 2\theta} \tag{5}$$

we see that,

$$p^2(\theta) + p^2(\theta + \pi/1) = 1.$$

The curvature of (c) is:

$$\frac{1}{k(\theta)} = \rho(\theta) = p(\theta) + \ddot{p}(\theta)$$

or

$$\rho(\theta) = \frac{1/4 - a^2}{[1/2 + a\cos 2\theta]^{3/2}}.$$

For $0 < a < 1/2$ it is $1/2 + a\cos 2\theta > 0$, therefore the curve (c) is convex. Also we can find the algebric equation of the curve (c). The point $M(x, y) \in (c)$ is given by the equations:

$$x = p(\theta)\cos\theta - \dot{p}(\theta)\sin(\theta)$$

$$y = p(\theta)\sin\theta + \dot{p}(\theta)\cos(\theta).$$

After some calculations we take:

$$\sin^2\theta = \frac{(1/2 + a)y^2}{(1/2 + a)^2 + 2ay^2}$$

$$\cos^2\theta = \frac{(1/2 - a)x^2}{(1/2 + a)^2 - 2ax^2}$$

and finally the equation of (c) is:

$$\frac{(1/2 + a)y^2}{(1/2 + a)^2 + 2ay^2} + \frac{(1/2 - a)x^2}{(1/2 + a)^2 - 2ax^2} = 1$$

that is (c) is a convex curve of 4^{th} degree.

3. Theorem (a): $A(K_*) \geq 2A(K)$.

Proof

From (3) follows:

$$x\dot{y} - y\dot{x} = p_1^2 + p_2^2 + p_1\dot{p}_2 + \dot{p}_1 p_2$$

hence,

$$A(K_*) = \frac{1}{2}\oint |x\dot{y} - y\dot{x}|$$

$$A(K_*) = \frac{1}{2}\int_0^{2\pi} p_1^2 d\theta + \frac{1}{2}\int_0^{2\pi} p_2^2 d\theta + \frac{1}{2}\int_0^{2\pi} (p_1 \dot{p_2}) d\theta$$

but,

$$\int_0^{2\pi} p_1 \dot{p_2} d\theta = 0$$

also,

$$\int_0^{2\pi} p_1^2 d\theta = \int_0^{2\pi} p_2^2 d\theta = 2A(K_0)$$

where K_0 the pedal curve of K, that is the locus of the feet of the perpendicular from the origin to the support line of K.
From the theorem of Chernoff, see [4], it is known that:

$$A(K) \leq \frac{1}{2}\int_0^{\pi/2} w(\theta) w(\theta + \pi/2) d\theta \tag{6}$$

where $w(\theta)$ is the width of K in the direction θ. But

$$w(\theta) = p(\theta) + p(\theta + \pi)$$

$$w(\theta + \pi/2) = p(\theta + \pi/2) + p(\theta + 3\pi/2).$$

From (6) follows:

$$A(K) \leq \frac{1}{2}\int_0^{2\pi} p(\theta) p(\theta + \pi/2) d\theta = \frac{1}{2}\int_0^{2\pi} p_1 p_2 d\theta$$

or,

$$A(K) \leq \frac{1}{4}\int_0^{2\pi} (p_1^2 + p_2^2) d\theta = A(K_0) = \frac{1}{2} A(K_*).$$

The equality from (6), for K=circle.

Theorem (b): $L(K_*) \geq \sqrt{2} L(K)$. Starting again from the relation (3), we take:

$$\dot{x}^2 + \dot{y}^2 = (p_1 + \dot{p_2})^2 + (p_2 - \dot{p_1})^2 \tag{7}$$

Using the well known inequality

$$A^2 + B^2 \geq \frac{1}{2}(A + B)^2$$

from (7) follows:

$$(p_1 + \dot{p_2})^2 + (p_2 - \dot{p_1})^2 \geq \frac{1}{2}\Big[|p_1 + \dot{p_2}| + |p_2 - \dot{p_1}|\Big]2$$

therefore,

$$L(K_*) = \int_0^{2\pi} (\dot{x}^2 + \dot{y}^2)^{\frac{1}{2}} d\theta = \frac{1}{\sqrt{2}} \int_0^{2\pi} (p_1 + \dot{p_2}) d\theta + \frac{1}{\sqrt{2}} \int_0^{2\pi} |p_2 - \dot{p_1}| d\theta$$

But,

$$\int_0^{2\pi} \dot{p_1} d\theta = 0, \quad \int_0^{2\pi} \dot{p_2} d\theta = 0$$

$$\int_0^{2\pi} |p_2 - \dot{p_1}| d\theta \geq |\int_0^{2\pi} (p_2 - \dot{p_1}) d\theta| = \int_0^{2\pi} p_2 d\theta$$

Also, it is:

$$\int_0^{2\pi} p_1 d\theta = \int_0^{2\pi} p_2 d\theta = L(K_*).$$

Hence

$$L(K_*) \geq \sqrt{2} L(K).$$

References

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