The problem (conjecture) of Larman-Zong

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Let $G = \{A_1, A_2, \dots\ ..A_{2d}\}$ be a point set in the $(d-1)$sphere $q = (O, 1) = S^{d-1} = \{x/|x| = 1\}$. We denote $P = \{x/A_i \cdot x \leq 1\}$ for $i = 1, 2, ..\ ..2d$. Assuming that the convex cover of G includes the center O of the sphere q, we are called to prove that there is a point $y \in P$ so that $|y| \geq \sqrt{d}$.
The general case of the above problem of Larman-Zong remains unsolved, for the last ten years. My solution for $d = 3$ is nearly elementary but unfortunatly does not work for $d > 3$. I have the information that Zong has the solution for $d = 3, 4$.
We start, for better understanding, the simple case for $d = 2$.
In the circle $q = (O, 1)$ we have four points A_1, A_2, A_3, A_4 so that the center O be in the interior of the quadrilateral $A_1A_2A_3A_4$. The tangent lines of the circle q at the points A_1, A_2, A_3, A_4 intersect and give the quadrilateral P. We will show that there is a vertex of P of a distance from O at least $\sqrt{2}$.
Let $w_1 = \angle A_1OA_2,\ w_2 = \angle A_2OA_3,\ \ w_3 = \angle A_3OA_4,\ \ w_4 = \angle A_4OA_1$. It is: $w_1 + w_2 + w_3 + w_4 = 2\pi$.
We suppose that w_2=max$[w_1, w_2, w_3, w_4]$.
Obviously $w_2 \geq \pi/2$.
In the triangle OA_2B_2, B_2 is the intersection of the tangents at the points A_2, A_3, so we have.

$$1 = OA_2 = OB_2 \cos w_2/2 \leq OB_2 \cos \pi/4 = OB_2 \frac{\sqrt{2}}{2}.$$

Or $OB_2 \geq \sqrt{2}$.
Proof *for* $\ d = 3$.
We suppose that the six points $A_1, A_2, ..\ ..A_6$ lie in the sphere $q = (O, 1)$ and are the vertices of a polyhedron A with 8 triangular facets. The polyhedron A includes the center O of the sphere q hence the tangent planes to q at the points $A_1, A_2, ..\ ..A_6$ are intersected at the edges of a polyhedron B with 6 faces and 8 vertices $B_1, B_2, ..\ ..B_8$.

We suppose now that A_0 is the regular polyhedron with 6 vertices $A_{01}, A_{02}, .. \ ..A_{06}$ (regular octahedron) and facets equilateral triangles inscribed to the sphere q. The tangent planes to q at the points $A_{01}, A_{02}, .. \ ..A_{06}$ are intersected and give the cube B_0 with vertices $B_{01}, B_{02}, .. \ ..B_{08}$. It is simple to see that:

$$OB_{0i} = \sqrt{3} \tag{1}$$

We consider the central projection of the regular octahedron A_0 from the point O to the sphere q. The result will be a regular spherical octahedron on q with facets the equilateral spherical triangles $S_{01}, S_{02}, .. \ ..S_{08}$.
Similarly, the cental projection from the point O of the octahedron A gives the spherical octahedron S with spherical facets $S_1, S_2, .. \ ..S_8$. We obviously have:

$$\cup_{i=1}^{8} S_i = \cup_{i=1}^{8} S_{0i} = 4\pi(surface\ q) \tag{2}$$

We assume now that $AreaS_k = max.AreaS_i$ for $i = 1, 2, ...8$. From (2) we conclude

$$AreaS_k \geq AreaS_{0i} \tag{3}$$

For $i = 1, 2, ...8$.
Let c_k the circumscribed circle of S_k on the sphere q and respectively c_0 the circle defined by S_{0i}. We denote by T_k the equilateral spherical triangle inscribed to the spherical circle c_k. It is well known that:

$$AreaT_k \geq AreaS_k \tag{4}$$

From (3) and (4) follows:

$$AreaT_k \geq AreaS_{0i}.$$

So we conclude that the circle c_k is no smaller than the circle c_0. Therefore the distance p_k of the center O from c_k is no bigger from the same elements of the regular octahedron respectively, that is:

$$p_k \leq p_{0k} = \frac{1}{\sqrt{3}} \tag{5}$$

But from the rectangular triangle OA_iB_i, we have:

$$OA_i^2 = 1 = p_k \cdot OB_i \leq \frac{1}{\sqrt{3}} OB_i$$

or,

$$OB_i \geq \sqrt{3}.$$

Unfortunately, it seems that the above proof, does not work for $d > 3$, so the problem is still open.