# Some Inequalities for a n-simplex

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(1). We recently proved an inequality between the width and the diameter of a n-simplex in $E^n$, see [3]. We continue this work using Leibniz formula and in the present note we give a stronger result including the width of a simplex and the sum of squares of its edges. We also have found an inequality joining the width and the circumradius as well as an inequality between the circumradius and the sum of the edges of a n-simplex.
Leibniz formula, or as it is better known, the polar moment of inertia theorem for a system of $k$ points $A_1, A_2, \ldots \quad ..A_k$ in $E^n$,with masses $m_1 m_2, \ldots \quad ..m_k$ respectively, asserts:

$$\sum_{i=1}^{k} \vec{A_i P}^2 = m\vec{GP}^2 + \frac{1}{m}\sum_{i<j}^{1,k} \vec{A_i A_J}^2$$

where $m = \sum_i^k m_i$, $G$ the center of mass of the system $A_1(m_1), A_2(m_2), .. \quad ..A_k(m_k)$ and $P$ some point in $E^n$, see[1],[2].
The above formula for $m_1 = m_2 = .... \quad m_k = 1$ and $G$ the centroid of the point set $\{A_1, A_2, \ldots \quad ..A_k\}$ gives:

$$\sum_{i=1}^{k} \vec{A_i P}^2 = k \cdot \vec{GP}^2 + \frac{1}{k}\sum_{i<j}^{1,k} \vec{A_i A_j}^2. \tag{1}$$

(2). Let now $A = \{A_1, A_2, .. \quad ..A_k\}$ and $B = \{B_1, B_2, .. \quad ..B_p\}$ two point sets in $E^n$ and $G_A, G_B$ their centroids respectively. We can prove the following generalization of the formula of Leibniz:

$$\sum_{j=1}^{k}\sum_{i=1}^{p} \vec{A_j B_i}^2 = kp\vec{G_A G_B}^2 + \frac{p}{k}\sum_{i<j}^{1,k} \vec{A_i A_j}^2 + \frac{k}{p}\sum_{i<j}^{1,p} \vec{B_i B_j}^2. \tag{2}$$

**Proof**
From formula (1) for $P = G_B$, we take:

$$\sum_{j=1}^{k} \vec{A_j G_B}^2 = k \cdot \vec{G_A G_B}^2 + \frac{1}{k} \sum_{q<l}^{1,k} \vec{A_q A_l}^2. \qquad (3)$$

The same formula (1) for the point system $B = \{B_1, B_2, \ldots \ldots B_p\}$ for $P = A_j$, gives:

$$\sum_{i=1}^{p} \vec{B_i A_j}^2 = p \cdot \vec{G_B A_j}^2 + \frac{1}{p} \sum_{s<t}^{1,p} \vec{B_s B_t}^2. \qquad (4)$$

We substitute $\vec{G_B A_j}^2$ from (4) in (3) and the summation with respect to $j$ from 1 to $k$ leads us to the formula (2).
(3). We can easily see that the number $T$ of the partitions of a point set $A = \{A_1, A_2, \ldots \ldots A_s\}$ into two subsets with cardinal numbers $k$ and $p$ , $k + p = s$, respectively is:
if $k = p$

$$T = T_1 = \frac{1}{2}\binom{k+p}{p}$$

for $k \neq p$

$$T = T_2 = \binom{k+p}{p}$$

Suppose that $G_k^r, G_p^r$ are the centroids of the two subsets of the $r^{th}$partition.We will calculate the sum:

$$G = \sum_{r=1}^{T} (\vec{G_k^r G_p^r})^2$$

(a). For $k = p$
In the formula (2) we put:
$A_j = A_{r_j}$, $B_i = A_{r_i}$ where: $r_i \neq r_j$

$$r_i \in \{1, 2, \ldots. \ , s\}$$

$$r_j \in \{1, 2, .. \ , s\}$$

That is $A_{r_j}$, $A_{r_i}$ belong to the first and the second subset of $A$ in the $r^{th}$partition. We apply formula (2) $T_1$ times (in each partition) and we add the resulting formulas. Putting

$$S = \sum_{i<j}^{1,s} \vec{A_i A_j}^2,$$

we see that in the first member $S$ appears

$$\frac{\frac{1}{2}T_1 kp}{\binom{p+k}{2}}$$

times.
From the last two terms of the second member of (2) $S$ appears (in the whole sum)

$$\frac{\frac{1}{2}T_1\left[\frac{p}{k}\binom{k}{2}+\frac{k}{p}\binom{p}{2}\right]}{\binom{p+k}{2}}$$

times.
Therefore we will have:

$$\frac{\frac{1}{2}\binom{p+k}{p}|\left[kp-\frac{p}{k}\binom{k}{2}-\frac{k}{p}\binom{p}{2}\right]S}{\binom{p+k}{2}}=pkG$$

or,

$$\frac{1}{2}\binom{k+p}{p}S=(k+p-1)kpG \tag{5}$$

(b). The formula for $k \neq p$ is similar. That is:

$$\binom{k+p}{p}S=(k+p-1)kpG. \tag{6}$$

From the formulas (5),(6) we easily see that

$$Minimum\ G$$

is taken for $k=[\frac{S}{2}]$, $k+p=S$.
(4). Assuming now that $S=n+1$ and $A_1A_2...\ \ ..A_{n+1}$ are the vertices of a n-simplex in $E^n$, formulas (5) and (6) give the minimum width of the simplex.
Indeed, we suppose $d$ be the minimum width of the simplex.
(a). Let $n$ be odd, hence $n+1$=even and $\frac{n+1}{2}$=integer. We consider a partition of the point set $A=\{A_1,A_2,...\ \ ..A_{n+1}\}$ into two parts having $\frac{n+1}{2}$ points in each part. Therefore each part determines an hyperplane parallel

to the other. We see that $d$ is no bigger than the distance of the Centroids of the two parts. So formula (5) finally gives:

$$S \geq n\Big[\frac{n+1}{2}\Big]^2 d^2 \tag{7}$$

(b). For $n$=even, we analogously take:

$$S \geq \frac{n^2(n+2)}{4} d^2 \tag{8}$$

From formulas (7) and (8) we find again our formulas in Crux, see [3]. Indeed it is enough to take:

$$\binom{n+1}{2} D^2 \geq S$$

where $D$ is the diameter of the simplex.
(5). From the formula of Leibniz (1), putting $P = O$ the circumcenter, we take:

$$(n+1)^2 R^2 \geq \sum_{i<j}^{1,n+1} \vec{A_i A_j}^2, \tag{9}$$

for the simplex $A_1 A_2 \ldots . A_{n+1}$.
For $n$ odd, from (7), (9) we take:

$$\frac{2R}{\sqrt{n}} \geq d \tag{10}$$

and for $n$ even from (8), (9) we take:

$$\frac{2(n+1)R}{n\sqrt{n+2}} \geq d \tag{11}$$

So we have an inequality between the width and the circumradius of the simplex.
(6). We can obtain another remarkable inequality between the circumradius and the sum of edges of a n-simplex, using again the formula of Leibniz. From (9) and the well known mean square inequality

$$\frac{1}{\binom{n+1}{2}} \sum_{i<j}^{1,n+1} \vec{A_i A_j}^2 \geq \left[\frac{\sum_{i<j}^{1,n+1} |\vec{A_i A_j}|}{\binom{n+1}{2}}\right]^2$$

we take:

$$R \geq \frac{S_1}{(n+1)\sqrt{\frac{n(n+1)}{2}}} \tag{12}$$

where, $S_1 = \sum_{i<j}^{1,n+1} |\vec{A_i A_j}|$. For $n = 2$ we have the well known triangle inequality:

$$3R\sqrt{3} \geq a + b + c$$

**References**

1. M.S.Klamkin,Geometric inequalities via the Polar Moment of Inertia, Math.Magazine 48 (1975),44-46.
2. G.Tsintsifas, Leibniz formula and Geometric Inequalities, in my Blog.
3. G.Tsintsifas, Crux Mathematicorum, problem 1162.