On Minkowski Geometry

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Introduction

Minkowski Geometry is an interesting realization of finite Banach spaces. In this note we extend some well known properties of the Euclidean space E^n in Minlowski space M^n.

Let O be the origin of the Euclidean space E^n and K a centrally symmetric compact convex body, x and y are points in E^n and $L(xy)$ the length of diameter of K in the direction of the line xy. The Minkowski distance is defined by:

$$\|x-y\| = \frac{2|x-y|}{L(xy)}$$

where $|x-y|$ is the Euclidean distance of the points x and y.

The proof that the above defined distance is a metric can be found in several places and for two different proofs the reader can consult [1] and [2]. The centrally symmetric convex body K will be refered as the unit Minkowskian ball and its boundary as the unit Minkowskian sphere and for simplicity usually it would be considered as a smooth surface.

Definations

Let $s^n = A_1A_2..\ \ ..A_{n+1}$ be a n-smplex in E^n, z_0 a point and R a real number so that s^n is inscribed in the convex body $K_0 = z_0 + R \cdot K$. We define K_0 as the circumscribed Minkowskian sphere of s^n and we denote it by $K_0(z_0, R)$. The point z_0 is the Minkowskian circumcenter and R the Minkowskian circumradius. We similarly define the Minkowskian insphere $K_i = z_i + r\dot{K}$ inscribed in s^n.

1. Generaliged Euler's inequality.

Let $K_0(z_0, R)$ the Minkowskian circumsphere and $K_i(z_i, r)$ the Minkowskian insphere of the n-simplex $s^n = A_1A_2\ldots\ \ldots A_{n+1}$ in M^n, then, it holds:

$$R \geq nr$$

see [4],[5].
Proof
Suppose B_i is the point of K_i on the facet s_i^{n-1} opposite to the vertex A_i and M an interior point in s^n. Let A_iH_i the Minkowskian altitude of s^n from the vertex A_i and M_i the point of intrsection of the parallel line from the M to the line z_iB_i with the plane s_i^{n-1}, that is the (Minkowskian) distance of the point M from s_i^{n-1}.
We will prove:

$$\sum_{i=1}^{n+1} q_i\|M - M_i\| = r \tag{1}$$

$$\sum_{i=1}^{n+1} q_i\|M - A_i\| \geq nr \tag{2}$$

where $q_1, q_2, \ldots .q_{n+1}$ the barycentric coordinates of the incenter z_i.
We easily see that:

$$q_i = \frac{\|z_i - B_i\|}{\|A_i - H_i\|}, \quad or \quad q_i = \frac{r}{h_i} \tag{3}$$

denoting $\|A_i - H_i\| = h_i$.
Also supposing that $m_1, m_2, \ldots\ \ldots m_{n+1}$ are the barycentric coordinates of the point M, we have:

$$\sum_{i=1}^{n+1} \frac{\|M - M_i\|}{h_i} = \sum_{i=1}^{n+1} m_i = 1$$

and using (3)

$$\sum_{i=1}^{n+1} q_i\|M - M_i\| = r.$$

The Minkowskian distance of the point A_i from the plane of s_i^{n-1} is h_i, so that it is obvious that:

$$\|A_i - M\| + \|M - M_i\| \geq h_i \tag{4}$$

or

$$q_i\|A_i - M\| + q_i\|M - M_i\| \geq q_ih_i = r$$

Therefore:
$$\sum_{i=1}^{n+1} q_i\|A_i - M\| + \sum_{i=1}^{n+1} q_i\|M - M_i\| \geq (n+1)r.$$
From the above proved relation (1) follows (2).
Euler's inequality between R and r is an easy application of (2). We take $M = z_0$ the center of the Minkowski sphere K_0. The equality in (2) follows from (4), that is if and only if M is the common point of the altitudes. So it will be $R = nr$ if and only if the circumcenter z_0 coincides with the common point of the altitudes.
Some other remarkable inequalities of the n-smplex can arise from the above: From the formula (3) we take
$$\sum_{i=1}^{n+1} \frac{1}{h_i} = \frac{1}{r} \tag{5}$$
and using Cauchy-Schwarz inequality we obtain
$$\sum_{i=1}^{n+1} h_i \geq (n+1)^2 \cdot r \tag{6}$$
Let now q_{ij}, $(i,j) \neq 1$ be the parallel line from the incenter z_i to the edge A_iA_j, P_i, P_j the common points of the insphere with q_{ij} and Q_i, Q_j the same points with the boundary of the simplex s^n (try with a tetrahedron $A_1A_2A_3A_4$).
We suppose that $A_1Q_i \cap A_iA_k = F_{ik}$ and $A_1Q_j \cap A_jA_k = F_{jk}$.
It is simple to see that:
$$1 - q_1 = \frac{\|Q_i - Q_j\|}{\|F_{il} - F_{jk}\|} > \frac{\|P_i - P_j\|}{\|A_i - A_j\|} = \frac{2r}{\|A_i - A_j\|}$$
or
$$\|A_i - A_j\| > \frac{2r}{1 - q_1}.$$
Repeating the same inequality for $q_2, q_3, ...q_{n+1}$ and adding we take:
$$\sum_{i>j}^{1,n+1} \|A_i - A_j\| > 2r \sum_{i=1}^{n+1} \frac{1}{1 - q_i}.$$
Last inequality and Cauchy-Schwarz inequality lead to:
$$\sum_{i>j}^{1,n+1} \|A_i - A_j\| > \frac{2r(n+1)^2}{n}. \tag{7}$$

For n=2 and an affine regular hexagon as the Minkowski unit sphere, the formula (7) can be the equality.
Another interesting Proof to the inequality of Euler is the following. Let A'_i be the centroid of the facet opposite to the vertex A_i of the simplex $s^n = A_1A_2...\ .A_{n+1}$. The simplices $s'^n = A'_1A'_2....\ .A'_{n+1}$ and s^n are similar with ratio $\frac{1}{n}$, hence, the radius of the Minkowskian sphere (z_g, R') the circumscribed to s'^n is $R' = R/n$. The sphere (z_g, R') is obviously no smaller than the inscribed sphere (z_i, r) in s^n, therefore $R/n \geq r$.

2. The problem of Fermat

G.D. Chakerian and M.A. Ghandehari have succefully investigated the well known problem of Fermat, see [7], in M^n. We will try on some aspect of the same problem.
We consider in M^n the simplex $s^n = P_1P_2...\ ..P_{n+1}$ and $m_1, m_2, ...\ ..m_{n+1}$ are positive real numbers so that $\sum_{i=1}^{n+1} m_i = 1$.
The problem is to determine the point x, so that the function

$$f(x) = \sum_{i=1}^{n+1} m_i \|x - P_i\|$$

has a minimum. Compactness arguments assure us that there is an interior point $z \in s^n$ so that:

$$f(z) = \sum_{i=1}^{n+1} m_i \|z - P_i\|$$

be a minimum.
Let now $K(z, 1)$ be the Minkowski unit sphere and B'_i the points of intersection of the lines zP_i with the sphere K.
The support planes at the points B'_i form a n-simplex $g^n = A'_1A'_2...\ ..A'_{n+1}$. We denote by g_i^{n-1} the $n-1$ simplex opposite the vertex A'_i, on the support plane at B'_i. From [7] page 230, we have:

$$gradf(z) = \sum_{i=1}^{n+1} m_i \frac{u_i}{p_i} = 0 \tag{8}$$

where, p_i the Euclidean distance of z from g_i^{n-1} and u_i is the unit vector perpendicular to the $n-1$ simplex g_i^{n-1} having an outword direction relative to g_i^{n-1}.
We also denote by V_i the volume of the simplex g_i^{n-1}. It is known that

$$\sum_{i=1}^{n+1} V_i u_i = 0 \tag{9}$$

The vectors $u_1, u_2, ... \; ..u_{n+1}$ are independant so from (8) and (9) follows:

$$\frac{m_1}{V_1 p_1} = \frac{m_2}{V_2 p_2} = ... \; .. = \frac{m_{n+1}}{V_{n+1} p_{n+1}}.$$

that is the incenter z of g^n must have as barycentric coordinates $m_1, m_2, ... \; ..m_{n+1}$.
We will prove now that, for every point $M \neq z$, $f(M) > f(z)$.
We consider from P_1 the parallel hyperplane to the facet g_i^{n-1}. These hyperplanes give a n-simplex $s^n = A_1 A_2 ... \; ..A_{n+1}$ having as insphere $K_i(z_i, r)$. The boundary points of K_i on its facets s_i^{n-1} are B_i.The simplices $A_1 A_2 .... A_{n+1}$ and $B_1 B_2 ..... B_{n+1}$ are similar therefore z and z_i have the same barycentric coordinates. Let M_i be the feet of the Minkowskian perpendicular from M to s_i^{n-1}. In (1) we have proved that:

$$\sum_{i=1}^{n+1} m_i \|M - M_i\| = r$$

and of course

$$\sum_{i=1}^{n+1} m_i \|z - P_i\| = r.$$

But, certenly holds:

$$\|M - P_i\| \geq \|M - M_i\|.$$

Therefore,

$$\sum_{i=1}^{n+1} m_i \|M - P_i\| > \sum_{I=1}^{n+1} m_i \|z - P_i\|$$

for $M \neq z$.

3. Generaliged Feuerbach circle.

We denote by G the centroid of the simplex $s^n = A_1 A_2 ... \; ..A_{n+1}$ and G_1 the centroid of the facet S_1^{n-1} opposite of the vertex A_1. The homothety F of center G and ratio $-1 : n$ transforms the Minkowski circumsphere $K_0(z_o, R)$ of the s^n int the sphere $K_f(z_f, R' = R : n)$. The point H determined from the relation

$$\vec{z_o H} = \sum_{i=1}^{n+1} \vec{z_o A_i} \tag{10}$$

is transformed (by F) to the point z_o and the sphere K_f intersects the line segments HA_i at the points D_i so that, $HD_i : HA_i = 1 : n$ That is because

$$\vec{z_o H} = \vec{z_o A_j} + \vec{A_j H}$$

and from (10)

$$\vec{z_oA_2} + \vec{z_oA_3} + ... + \vec{z_oA_{n+1}} = \vec{A_1H}$$

or

$$n \cdot \vec{z_oG_1} = \vec{A_1H}$$

For n=2 and K_o the circumcircle of the triangle $A_1A_2A_3$ we recognize K_f as the Feuerbach circle or the circle of the nine points.
The following remarks are very simple.
1. If z_oG_i is prpendicular to the facet s_i^{n-1}, for $i = 1, 2, 3.....n+1$, the altitudes of the smplex s^n have a common point.
2. Suppose that the altitudes of the simplex s^n have a common point H and

$$\vec{z_oH} = \sum_{i=1}^{n+1} z)\vec{o}A_i$$

then the line z_oG_i is perpendicular to the facet s_i^{n-1}.
3. In M^2 we consider the Minkowski unit circle as ellipse. Therefore K_f is an ellipse. An affine transformation transforms K_f to a circle c and informs us that K_f contains:
(a). The middle points of the sides of the triangle $A_1A_2A_3$.
(b). The middle points of the line segments HA_1, HA_2, HA_3, where HA_i is parallel to the line z_oG_i.
(c). The points of intersections of the lines HA_1, HA_2, HA_3 with the sides A_2A_3, A_3A_1, A_1A_2, respectively.

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