Problem (The inscircle in a regular tetrahedron)

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The inscircle of a tetrahedron is a circle of maximum radius inscribed in the tetrahedron for every direction in E^3. Find the radius of the inscircle of the regular tetrahedron.

Solution.

Let $s = A_1A_2A_3A_4$ be a regular tetrahedron and p be a plane of direction t_0 in E^3. We denote by $w = p \cap s$ and let r_1 be the radius of the inscircle in w.
We suppose:
(1). $r_{t_0} = max.r_1$.
(2). $r_t = min.r_{t_0}$, for every direction t_0. We are looking for r_t. That is the radius of the inscircle of s is r_t.
We need the following lemma.
Lemma
Let $O\vec{M}_1 = u_1,\ O\vec{M}_2 = u_2,\ O\vec{M}_3 = u_3,\ O\vec{M}_4 = u_4$ four unit vectors so that $M_1M_2M_3M_4$ be a regular tetrahedron, then for every unit vector v holds:

$$\sum_1^4 (u_i \cdot v)^2 = \frac{4}{3}.$$

Proof
We use barycentric coordinates, We can find the numbers m_1, m_2, m_3, m_4 so that: $v = \sum_1^4 m_i u_i$ and $\sum_1^4 m_i = 1$.
But $\sum_1^4 u_i = 0$ and $|u_i| = 1$. We easily find $(u_i, u_j) = -1/3$.

So we will have:

$$(a) \qquad (u_i, v) = m_j - \frac{1}{3}(1 - m_j), \quad or \;\; m_j = \frac{1 + 3(u_i, v)}{4}$$

But

$$v^2 = (\sum_1^4 m_i \cdot u_i)^2 = \sum_1^4 m_i^2 - 2/3 \sum_{i \geq j}^{1,4} m_i m_j = 1$$

Taking in mind that $\sum_1^4 m_i = 1$, we easily find $\sum_1^4 m_i^2 = 1$.
Squaring (a) and adding from 1 to 4 we take:

$$\sum_1^4 (u_i, v)^2 = 4/3$$

Now we go back to the original problem.
We suppose that s is circumscribed to the unit sphere $(O, 1)$, $Ov_1v_2v_3$ be an orthonormal Cartesian system and $v_3 = t_0$.
The equation of the inscribed circle (q) in w is:

$$(q): \quad (x - y, v_1)^2 + (x - y, v_2)^2 = r_1^2 \tag{1}$$

where y is the center of (q) and $x \in (q)$.
We denote by u_1, u_2, u_3, u_4 the unit vectors perpendicular to the facets of s and with direction outside of s. Obviously $\sum_1^4 u_i = 0$.
The point

$$x_i = y + r \frac{(u_i, v_1)v_1 + (u_i, v_2)v_2}{\left[\,|\, (u_i, v_1)v_1 + (u_i, v_2)v_2 \,|\, \right]^{1/2}} \tag{2}$$

lies on the circle (q).
The formula (2) can be written:

$$x_i = y + \frac{Pu_i}{|Pu_i|}, \tag{3}$$

where $Pu_i = (u_i, v_1)v_1 + (u_i, v_2)v_2$.
The point x_i is an interior point of s, so we will have:

$$(u_i, x_i) \leq 1,$$

or

$$(y + r_1 \frac{Pu_i}{|Pu_i|}, u_i) \leq 1,$$

or

$$(y, u_i) + \frac{r_1(Pu_i, u_i)}{|Pu_i|} \leq 1. \tag{4}$$

But

$$(Pu_i, u_i) = (u_i, v_1)^2 + (u_i, v_2)^2 = |Pu_i|^2,$$

and from (4) follows:

$$(y, u_i) + r_1|Pu_i| \leq 1. \tag{5}$$

The equality in (5) holds for $r_1 = r_{t_0}$,that is, when the incircle has points of contact in every facet of s. Hence,

$$(y, u_i) + r_{t_0}|Pu_i| = 1. \tag{6}$$

From (6) follows:

$$(y, u_1 + u_2 + u_3 + u_4) + r_{t_0} \sum_1^4 |Pui| = 4,$$

hence

$$r_{t_0} = \frac{4}{\sum_1^4 |Pu_i|} \ . \tag{7}$$

From (7) we easily understand that:

$$r_{t0} = \frac{4}{max \sum_1^4 |Pu_i|} \ , \tag{8}$$

so, we will try to deternine the $max \sum_1^4 |Pu_i|$.
Obviously

$$1 = (u_i, v_1)^2 + (u_i, v_2)^2 + (u_i, v_3)^2 = (u_i, v_3)^2 + |Pu_i|^2,$$

Therefore,

$$4 = \sum_1^4 (u_i, v_3)^2 + \sum_1^4 |Pu_i|^2.$$

From the lemma we know that $\sum_1^4 (u_i, v_3)^2 = 4/3$, so we have:

$$\sum_1^4 |Pu_i|^2 = 8/3. \tag{9}$$

Caushy-Schwarz inequality asserts:

$$\frac{\sum_1^4 |Pu_i|^2}{4} \geq \left[\frac{\sum_1^4 |Pu_i|}{4}\right]^2$$

or,

$$4\sqrt{\frac{2}{3}} \geq \sum_1^4 |Pu_i|$$

and from (8) follows:

$$r_t = \sqrt{\frac{3}{2}}. \tag{10}$$

The equality holds when

$$|Pu_1| = |Pu_2| = |Pu_3| = |Pu_4|$$

or,

$$(u_1, v_3)^2 = (u_2, v_3)^2 = (u_3, v_3)^2 = (u_4, v_3)^2. \tag{11}$$

The existance

The Analytical and Logical steps lead us to the formulas (10) and (11) but we have now to prove that there is a vector v_3 and an interior to s circle of radius $\sqrt{3/2}$, so that (10) and (11) be correct.

Indeed, let be the regular tetrahedron $s = ABCD$ circuscribed to the sphere (O,1) and K, L, M, N the middle points of the edges AB, AC, CD, BD. The inscribed circle in the square $KLMN$ has radius $r_1 = \sqrt{3/2}$. Therefore the circle $(O, \sqrt{3/2})$ is the max. inscribed circle in s. This circle can rotate inside to s. Also the unit vector perpendicular to the plane $KLMN$ can be taken as the vector v_3 in the formulas (11).