Some Inequalities for a triangle.

G.Tsintsifas

Let ABC be a triangle and M an interior point. We denote by R_m the sum of the distances of the point M from the vertices of ABC and by r_m the sum of the distances of the point M from the sides.
We will prove:

$$(\mathbf{a}). \qquad r_O \geq r_G \geq r_I \geq r_H \tag{1}$$

$$(\mathbf{b}). \qquad R_O \geq R_H \geq R_G \geq R_I \tag{2}$$

Where by O, G, I, H are respectively denoted the pericenter, the centroid, the incenter and the orthocenter.

Proof

(1). $r_O \geq r_G$
We obviusly have

$$3r_G = h_1 + h_2 + h_3$$

but

$$h_1 + h_2 + h_3 \leq 5r + 2R \leq 3(R + r).$$

Also $3r_O = 3(R + r)$,
Therefore: $r_O \geq r_G$, see [1], 6.12

(2). $r_G \geq r_I$.

$$3r_G = h_1 + h_2 + h_3 \geq 9r$$

hence $r_G \geq r_I$,
see[1], 7.12

(3). $r_I \geq r_H$

$$2R + 5r \geq h_1 + h_2 + h_3 = r_H + 2(R + r)$$

that is $r_I \geq r_H$, see [1], 6.12

(b).
(1). $R_O \geq R_H$
$R_H = 2(p_1 + p_2 + p_3$ where p_1, p_2, p_3 are the distances of the pericenter from the sides of the triangle ABC, that is $r_O = p_1 + p_2 + p_3 = R + r$, hence,

$$R_H = 2(R + r0) \leq 3R = R_O$$

(2). $R_H \geq R_G$
We have

$$R_H = 2(R + r) \geq 3R = R_O$$

according [1], 6.2

(3). $R_G \geq R_I$

Let e_1, e_2, e_3 be the unit vectors of $\vec{IA}, \vec{IB}, \vec{IC}$. We will have:

$$\vec{IM} = e_1 + e_2 + e_3 = \frac{\vec{IA}}{IA} + \frac{\vec{IB}}{IB} + \frac{\vec{IC}}{IC} \tag{3}$$

Or

$$\vec{IM} = \frac{1}{r}(\sin\frac{A}{2} \cdot \vec{IA} + \sin\frac{B}{2} \cdot \vec{IB} + \sin\frac{C}{2} \cdot \vec{IC}) \tag{4}$$

that is the barycentric coordinates of the point M are:

$$\lambda \sin\frac{A}{2},\ \lambda \sin\frac{B}{2},\ \lambda \sin\frac{C}{2},$$

where

$$\lambda = \frac{1}{\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}}.$$

Also $\vec{IG} = \frac{1}{3}(\vec{IA} + \vec{IB} + \vec{IC})$, that is the barycentric coordinates of the point G are 1/3,1/3,1/3.
We can now suppose that $a \geq b \geq c$ and AD, BE, CZ the bissectrices from A, B, C. We easily determine the position of the points M and G inside the triangle ABC, that is: The point M is in the triangle AIZ and the point G in the triangle GID. Therefore $\angle GIM > \angle DIZ = \pi/2 + B/2$, hence

$$\vec{IG} \cdot \vec{IM} \leq 0. \tag{5}$$

We now have

$$IA + IB + IC = \vec{IA} \cdot e_1 + \vec{IB} \cdot e_2 + \vec{IC} \cdot e_3 =$$

$$= (\vec{IG} + \vec{GA})e_1 + (\vec{IG} + \vec{GB})e_2 + (\vec{IG} + \vec{GC})e_3 =$$

$$= \vec{IG}(e_1 + e_2 + e_3) + \vec{GA} \cdot e_1 + \vec{GB} \cdot e_2 + \vec{GC} \cdot e_3$$

That is:

$$IA + IB + IC = \vec{IG} \cdot \vec{IM} + \vec{GA} \cdot e_1 + \vec{GB} \cdot e_2 + \vec{GC} \cdot e_3 \tag{6}$$

From (5),(6) follows.

$$IA + IB + IC \leq |\vec{GA} \cdot e_1| + |\vec{GB} \cdot e_2| + |\vec{GC} \cdot e_3| \leq GA + GB + GC$$

.

remark 1. The proof of (b).3 is based in the very clever solution of my friend Nikos Deriades who first proved (5).
remark 2. Another, probably normal way, to solve the problem is the following.
We drop the perpendiculars BB', CC' and from the middlepoint M of BC to the bissectrice AD. The following relations are very easy.

$$2AM \geq 2AM' = 2AI + IB' + IC' = 2AI + IB\sin\frac{C}{2} + IC\sin\frac{B}{2}.$$

Therefore we have to prove:

$$\sum (IB \sin \frac{C}{2} + IC \sin \frac{B}{2}) \geq IA + IB + IC. \tag{7}$$

We know that $r = IA \sin \frac{A}{2},\ r = IB \sin \frac{B}{2},\ r = IC \sin \frac{C}{2}$. After the subtitution in the above inequality we find a trigonometric inequality, but it is quite difficult to prove it. The proof of (7) can follow easily from (5) because (7) is equivalent to the following

$$(\vec{IA} \sin \frac{A}{2} + \vec{IB} \sin \frac{B}{2} + \vec{IC} \sin \frac{C}{2})(\vec{IA} + \vec{IB} + \vec{IC}) \leq 0.$$

References

1. O.Bottema, R.Z.Djordjevic, R.R.Janic, D.S.Mitrinovic, P.M.Vasic, Geometric Inequalities, Wolters-Noordhoff