Inequalities for a simplex and a triangle.

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Problem .

Let $S_n = A_1A_2..\ ..A_{n+1}$ be a n-simplex in E^n and M is an interior point. The line A_iM intersects the opposite face, that is the simplex $S^i_{n-1} = A_1A_2...A_{i-1}A_{i+1}....A_{n+1}$ at the point A'_i and the simplex $S'^i_{n-1} = A'_1A'_2...A'_{i-1}A'_{i+1}...\ A'_{n+1}$ at the point A''_i. Prove:

$$\sum_1^{n+1} \frac{A_iA''_i}{A'_iA''_i} \geq n^2 - 1. \tag{1}$$

Proof.

We denote, as useally, the vector of position of a point Q by $\vec{OQ} = Q$. Therefore the point M expressed by its barycentric coordinates , is:

$$M = \sum_1^{n+1} q_iA_i \ \ where \ \ \sum_1^{n+1} q_i = 1, \ \ q_i \geq 0.$$

So, we have:

$$M = q_iA_i + (1 - q_i)\frac{\sum_{\substack{j=1\\ j\neq i}}^{n+1} q_jA_j}{1 - q_i}$$

That is:

$$A'_i = \frac{\sum_{\substack{j=1\\ j\neq i}}^{n+1} q_jA_j}{1 - q_i} \tag{2}$$

Similarly:
$M = \sum_1^{n+1} m_iA'_i, \ \ \sum_i^{n+1} = 1, \ \ m_i \geq 0$, or

$$M = \sum_{i=1}^{n+1} m_i \left[\sum_{\substack{j-1\\ j\neq i}}^{n+1} \frac{q_jA_j}{1 - q_i} \right] = \sum_{i=1}^{n+1} q_iA_i,$$

So we take: $m_i = \frac{1-q_i}{n}$. Indeed, we have:

$$\frac{MA''_i}{A'_iA''_i} = m_i,\ (a) \qquad \frac{MA'_i}{A_iA'_i} = q_i,\ (b)$$

from (b) follows that:

$$\frac{A'_iA''_i - MA''_i}{A_iA''_i + A'_iA''_i} = q_i \Rightarrow$$

$$\frac{1 - \frac{MA''_i}{A'_iA''_i}}{1 + \frac{A_iA''_i}{A'_iA''_i}} = q_i \Rightarrow$$

$$\frac{1 - m_i}{1 + \frac{A_iA''_i}{A'_iA''_i}} = q_i \Rightarrow$$

$$\frac{A_iA''_i}{A'_iA''_i} = \frac{1 - m_i - q_i}{q_i}$$

but we proved in problem 1 that $m_i = \frac{1-q_i}{n}$, therefore

$$\frac{A_iA''_i}{A'_iA''_i} = \frac{(n-1)(1-q_i)}{nq_i} = \frac{(n-1)}{n}\Big[\frac{q_1 + q_2 + ..q_{i-1} + q_{i+1}..q_{n+1}}{q_i}\Big]$$

and finally,

$$\sum_{1}^{n+1} \frac{A_iA''_i}{A'_iA''_i} = \frac{n-1}{n} \sum_{i \geq j}^{1,n+1} \Big[\frac{q_i}{q_j} + \frac{q_j}{q_i}\Big] \geq n^2 - 1.$$

The above is the conjecture 7.19 page 338 in [1] (for a triangle).

Some applications for a triangle.

For n=3, that is for the triangle $A_1A_2A_3$ using the above formulas we can obtain some remarkable results.

1.The points $A'_1, A''_1 - M, A_1$ are Harmonic range, that is:$\frac{MA''_1}{MA"'_1} = \frac{A_1A''_1}{A_!A'_1}$ (c)

Proof

From (a) follows.

$$m_1 = \frac{MA''_1}{MA''_1 + MA'_1}$$

or equivalently for (c)

$$m_1 = \frac{A_1A''_1}{A_1A''_1 + A_1A'_1}$$

$$\Rightarrow m_1 = \frac{1}{1+\frac{A_1A_1'}{A_1A_1''}} = \frac{1}{1+\frac{A_1A_1''+A_1''A_1'}{A_1A_1''}}$$

$$\Rightarrow m_1 = \frac{1}{2+\frac{q_1}{1-m_1-q_1}} = 1-m_1-q_1 = 1-\frac{1-q_1}{2}-q_1 = \frac{1-q_1}{2}$$

That is according (a),(b) we conclude that $A_1', A_1'' - M, A_1$ are Harmonic range.

2.

$$\sum_{(cyclic)} [A_2A_1''A_3]^2 \leq [A_1A_2A_3]^2$$

Proof.

Descarte's theorem for the harmonic ranges is:

$$\frac{2}{A_1'A_1''} = \frac{1}{A_1'M} + \frac{1}{A_1A_1'}$$

$$\Rightarrow A_1'A_1'' = \frac{2MA_1' \cdot A_1A_1'}{MA_1' + A_1A_1'} \leq \sqrt{MA_1' \cdot A_1A_1'}$$

$$\Rightarrow \frac{A_1'A_1''}{MA_1'} \leq \frac{A_1A_1'}{A_1'A_1''} \Rightarrow \frac{(A_2A_1''A_3)}{(A_2MA_3)} \leq \frac{(A_1A_2A_3)}{(A_2A_1''A_3)}$$

$$(A_2A_1''A_3)^2 \leq (A_1A_2A_3) \cdot (A_2MA_3)$$

and finally

$$\sum_{(cyclic)} (A_2A_1''A_3)^2 \leq (A_1A_2A_3)^2$$

3.

$$\sum_{(cyclic)} (A_2'A_1A_3') \geq 3(A_1'A_2'A_3')$$

Proof.

From (1), fof $n = 3$ follows.

$$\sum_{(cyclic)} \frac{A_1A_1''}{A_1''A_1'} \geq 3 \ \Rightarrow \sum_{(cyclic)} \frac{(A_2'A_1A_3')}{(A_1'A_2'A_3')} \geq 3$$

$$\Rightarrow \frac{(A_1A_2A_3) - (A_1'A_2'A_3')}{(A_1'A_2'A_3')} \geq 3$$

or

$$\sum_{(cyclic)} (A_2'A_1A_3') \geq 3(A_1'A_2'A_3')$$

and from the above

$$max[(A_2'A_1A_3'),(A_3'A_2A_1'),(A_1'A_3A_2')] \geq (A_1''A_2'A_3') \quad (d)$$

$$(A_1A_2A_3) \geq 4(A_1'A_2'A_3') \quad (e)$$

References.

1. Recent Advances in Geometric Inequalities, D.S.Mitrinovic, J.E.Pecaric, and V.Volonec, Kluwer Academic Publishers.