Some Geometrc and Analytic Inequalities.

G.Tsintsifas

Introduction.

The most part of this paper are results obtained since 1983. The starting point was the following Geometric problem.

Problem 1.

Let $S_n = A_1A_2..\ ..A_{n+1}$ be a n-simplex in E^n and M is an interior point. The line A_iM intersects the opposite face, that is the simplex $S^i_{n-1} = A_1A_2...A_{i-1}A_{i+1}....A_{n+1}$ at the point A'_i and the simplex $S'^i_{n-1} = A'_1A'_2...A'_{i-1}A'_{i+1}...\ A'_{n+1}$ at the point A''_i. We denote by V, V', V'' the volumes of the simplices $S_n,\ S'_n = A'_1A'_2...\ \ A'_{n+1}$ *and* $S''_n = A''_1A''_2...\ \ A''_{n+1}$. Then it holds:

$$V \cdot V' \geq (V')^2. \tag{1}$$

Problem 2.

Working with barycentric coordinates we were lead to the following inequality.

For $x_1, x_2, ...\ x_n$ positive numbers and $S = x_1 + x_2 + ..\ ..x_n$ it holds:

$$\prod_{i=1}^{n}(S - x_i)^2 \geq \prod_{i=1}^{n} x_i[(n-2)S + x_i] \tag{2}$$

Problem 3.

We also found and some others interesting inequalities.

$$x_1x_2....x_n\left[\frac{S}{n}\right]^n \leq \left[\frac{S = x_1}{n-1}\right]^2\left[\frac{S - x_2}{n-1}\right]^2....\left[\frac{S - x_n}{n-1}\right]^2. \tag{3}$$

Problem 4.

For the convex function $f(x)$ holds:

$$\sum_{i=1}^{n} f(x_i) + nf\Big(\frac{S}{n}\Big) \geq 2f\sum_{i=1}^{n}\Big(\frac{S - x_i}{n-1}\Big) \tag{4}$$

Problem 5. The following is very interesting but I do not have a proof for the general case (a)

$$T_r = \prod_{i_1<i_2<...<i_{n-r}}^{1\le i_t\le n-1} \left[\frac{x_{i1}+x_{i2}+....+x_{i(n-r)}}{n-r}\right]^{\frac{1}{\binom{n}{n-r}}} \quad (5)$$

Prove:

$$(a): \quad T_r^2 \ge T_{r-1}T_{r+1} \quad (6)$$

$$(b): \quad T_r \ge T_{r+1} \quad (7)$$

The main tools we have used are:
(1). The inequality of Popoviciu, see [4], and
(2). the Majorization theory, see [2],[3] and [4].
The inequality of Popoviciu is:

$$\binom{n-2}{k-2}\left[\frac{n-k}{k-1}\sum_{i=1}^{n} p_i f(x_i) + \Big(\sum_{i=1}^{n} p_i\Big) f\left(\frac{\sum_i^n p_i x_i}{\sum_i^n p_i}\right)\right] \quad (8)$$

$$\ge \sum (p_{i1}+p_{i1}+...+p_{ik}) f\left(\frac{p_{i1}x_{i1}+p_{i2}x_{i2}+...+p_{ik}x_{ik}}{p_{i1}+p_{i2}+...+p_{ik}}\right).$$

For $1 \le i_1 \le i_2 \le .... \le i_k \le n,\ p_i \ge 0,\ f(x)$ is convex function.
The basis of the Majorization method is the excelent theorem of Hardy-Littlewood-Polya.
Let $x = [x_1\ x_2\ ....\ x_n],\ \ y = [y_1\ y_2\ ....\ y_n]$ be two points in E^n so that:

$$x_1 \le x_2 \le .... \quad \le x_n$$

$$y_1 \le y_2 \le .... \quad \le y_n$$

. The following conditions are equivalent:
(1). There is a bistohastic matrix Λ so that:

$$y^T = \Lambda x^T$$

(2). There is a convex function f, so that:

$$\sum_1^n f(x_i) \ge \sum_1^n f(y_i).$$

(3).

$$x_1 \le y_1$$
$$x_1 + x_2 \le y_1 + y_2$$
$$x_1 + x_2 + x_3 \le y_1 + y_2 + y_3$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x_1 + x_2 + \ldots.x_{n-1} \le y_1 + y_2 + \ldots.y_{n-1}$$
$$x_1 + x_2 + \ldots.x_n = y_1 + y_2 + \ldots.y_n.$$

The condition (3) is denoted by: $(x_1, x_2, \ldots.x_n) \prec (y_1, y_2, \ldots.y_n)$.
See for the proof in [2].
3(a). The majorization theorem is equivalent to the following. If

$$x_1 \ge x_2 \ge \ldots. \quad \ge x_n$$
$$y_1 \ge y_2 \le \ldots. \quad \ge y_n$$

. and $(x_1, x_2, \ldots.x_n) \succ (y_1, y_2, \ldots.y_n)$. Then it holds (2).

Proofs
We start the **proof of (1)**.
We denote, as useally, the vector of position of a point Q by $\vec{OQ} = Q$. Therefore the point M expressed by its barycentric coordinates , is:

$$M = \sum_1^{n+1} q_i A_i \;\; where \;\; \sum_1^{n+1} q_i = 1, \;\; q_i \ge 0.$$

So, we have:

$$M = q_i A_i + (1 - q_i)\frac{\sum_{\substack{j=1\\ j\neq i}}^{n+1} q_j A_j}{1 - q_i}$$

That is:

$$A'_i = \frac{\sum_{\substack{j=1\\ j\neq i}}^{n+1} q_j A_j}{1 - q_i} \tag{9}$$

Similarly:
$M = \sum_1^{n+1} m_i A'_i, \;\; \sum_i^{n+1} = 1, \;\; m_i \ge 0$, or

$$M = \sum_{i=1}^{n+1} m_i \left[\sum_{\substack{j-1\\ j\neq i}}^{n+1} \frac{q_j A_j}{1 - q_i} \right] = \sum_{i=1}^{n+1} q_i A_i,$$

So we take: $m_i = \frac{1-q_i}{n}$. From (9) we see that:

$$V' = |\Lambda| V$$

where Λ is the matrix of the transfomation the simplex S_n to S'_n. After the calculations we take:

$$V' = n \prod_{i=1}^{n+1} \frac{q_i}{1-q_i} V \tag{10}$$

Similarly we take:

$$V'' = n^2 \prod_{i=1}^{n+1} \frac{q_i}{n-(1-q_i)} V \tag{11}$$

So, we easily see that we are leading to the inequality (2).

The proof of the inequality (2)

By the A.M-G.M inequality follows:

$$S - x_i \geq (n-1) \prod_{\substack{j=1 \\ j \neq i}}^{n} x_j^{\frac{1}{n-1}} \tag{12}$$

From (12) we have:

$$\prod_1^n (S - x_i) \geq (n-1)^n \prod_1^n x_i$$

or,

$$\frac{\prod_1^n (S-x_i)}{[(n-1)S]^n} \geq \frac{\prod_1^n x_i}{[\sum_1^n x_i]^n}$$

$$\frac{\prod_1^n (S-x_i)}{[\sum_1^n (S-x_i)]^n} \geq \frac{\prod_1^n x_i}{[\sum_1^n x_i]^n} \tag{13}$$

From (13),we take:

$$\frac{\prod_{\substack{i=1 \\ i \neq j}}^n (S-x_i)}{\left[\sum_{\substack{i=1 \\ i \neq j}}^n (S-x_i)\right]^n} \geq \frac{\prod_1^n x_i}{\left[\sum_{\substack{i=1 \\ i \neq j}}^n x_i\right]^n} \tag{14}$$

and from the above (14) we go to the next

$$\prod_{i=1}^n \frac{\prod_{\substack{j=1 \\ j \neq i}}^n (S-x_j)}{[(n-2)S + x_i]^{n-1}} \geq \prod_{i=1}^n \frac{\prod_{\substack{j=1 \\ j \neq i}}^n x_j}{[S - x_i]^n}$$

because of

$$(n-2)S+x_i=\sum_{\substack{j=1\\ j\neq i}}^{n}(S-x_j).$$

Therefore

$$\frac{\prod_1^n (S-x_i)^n}{\prod_1^n \Big[(n-2)S+x_i\Big]^n}\geq \frac{\prod_1^n x_i^{n-1}}{\prod_1^n \Big[S-x_i\Big]^{n-1}}$$

and finally:

$$\prod_1^n (S-x_i)^2\geq \prod_1^n x_i[(n-2)S+x_i].$$

Proof of the inequality (4)
Follows from Popoviciu inequality (3).
Indeed, we have for $k=n-1$, and $p_i=1$

$$f(x_1)+f(x_2)+....f(x_n)+n(n-2)f\Big(\frac{x_1+x_2+....x_n}{n}\Big)$$

$$\geq (n-1)\Bigg[f\Big(\frac{S-x_1}{n-1}\Big)+.....+f\Big(\frac{S-x_n}{n-1}\Big)\Bigg]$$

Therefore we only have to prove:

$$(n-1)\sum f\Big(\frac{S-x_i}{n-1}\Big)-n(n-2)f\Big(\frac{S}{n}\Big)\geq 2\sum f\Big(\frac{S-x_i}{n-1}\Big)-nf\Big(\frac{S}{n}\Big)$$

or equivalently

$$\sum f\Big(\frac{S-x_i}{n-1}\Big)\geq nf\Big(\frac{S}{n}\Big)$$

which follows directly from the Jensen inequality, that is:

$$\frac{1}{n}\sum f\Big(\frac{S-x_i}{n-1}\Big)\geq f\Big(\sum \frac{S-x_i}{n(n-1)}\Big)=f\Big(\frac{S}{n}\Big).$$

The proof of the inequality (3)
It follows from (4),taking as $y=f(x)$ the convex function $y=-Lnx$. That is:

$$Lnx_1+Lnx_2+....Lnx_n+nLn\frac{S}{n}\leq \sum Ln\Big(\frac{S-x_i}{n-1}\Big)^2$$

or,

$$x_1x_2....x_n\Big(\frac{S}{n}\Big)^n \leq \prod_1^n \Big(\frac{s-x_i}{n-1}\Big)^2.$$

It is worthwile to notice that (2),(3) and (4) can be proven using the majorization method. Also Popoviciu inequality for $k=n-2$, $p_i=1$ gives:

$$\sum_1^n f(x_i) + \frac{n(n-3)}{2} f\Big(\frac{S}{n}\Big) \geq \sum_{\substack{i>j\\ i,j}}^{1,n} f\Big(\frac{S-x_i-x_j}{n-2}\Big) \tag{15}$$

and the above (15) leads to the next

$$x_1x_2....x_n\Big(\frac{S}{n}\Big)^{\frac{n(n-3)}{2}} \leq \prod_{\substack{i>j\\ i,j}}^{1,n} \Big(\frac{S-x_i-x_j}{n-2}\Big). \tag{16}$$

For (5) it is easy to prove that $T_k \geq T_{k+1}$.
Also $T^2 \geq T_0T_2$, or equivalently

$$\Big[\frac{(x_1+x_2)}{2}\frac{(x_2+x_3)}{2}\frac{(x_3+x_1)}{2}\Big]^2 \geq x_! x_2 x3 \Big[\frac{x_1+x_2+x_3}{3}\Big]^3 (17)$$

A remarkable function
The function

$$f(x) = \frac{g(x)}{w(x)}$$

where

$$g(x) = (x+x_1)^{a_1}(x+x_2^{a_2}....(x+x_n)^{a_n}$$
$$w(x) = [a_1(x+x_1)+a_2(x+x_2)+....a_n(x+x_n)]^{s_a}$$

$s_a = a_1+a_2+.....a_n, \quad x_i, a_i \in \Re^+$
We see that $\frac{g'(x)}{g(x)} = \sum \frac{x_i}{x+x_i}$, so we will have:

$$f'(x) = \frac{g(x)}{[w(x)]^{s_a-1}}\Big[(\sum \frac{a_i}{a+x_i})(\sum a_i(x+x_i)) - s_a^2\Big]$$

From Gauchy-Schwarz inequality, we see that for $x \geq 0, \quad f'(x) > 0$.
Therefore $f(x)$ is increasing it is easy to see $f(x) < 1$ that is $f(x)$ is bounded above, hence $f(x)$ has a limit:

$$x \to \infty : \quad f(x) \to \frac{1}{s_a^{s_a}}$$

hence,

$$\frac{1}{s_a^{s_a}} \geq f(x) = \frac{g(x)}{w(x)} \geq \frac{g(0)}{w(0)}$$

or

$$a_1x_1 + a_2x_2 + ... + a_nx_n \geq s_a \sqrt[s_a]{x_1^{a_1}x_2^{a_2}...x_n^{a_n}} \qquad (18)$$

For $a_1 = a_2 = ...a_n = 1$, (18) is the well known A.M.-G.M. inequality.
Many remarkable inequalities follow from (18), e.g.

$$\Big[\frac{ax+by}{a+b}\Big]^{a+b} \geq x^a y^b$$

or, the Young's inequality, for $\frac{1}{a} + \frac{1}{b} = 1$

$$\frac{x^a}{a} + \frac{y^b}{b} \geq xy$$

The function $f_1(x)$
where

$$f_1(x) = \frac{g_1(x)}{w_1(x)},\ g_1(x) = \prod_1^n (x - x_i),\ w_1(x) = \Big[\sum_1^n (x - x_i)\Big]^n,$$

$x = \sum_1^n x_i,\ \ x_i \in \Re^+,\ S = \sum_1^n x_i$.
Working as previously we see that $f_1(x)$ is increasing. We also can prove that $f_1(x) \geq f_1(0)$. That is because:

$$S - x_j \geq (n-1)\Big[\prod_{\substack{i \neq j \\ i=1}}^n x_i\Big]^{\frac{1}{n-1}}$$

We multiplay the n-1 above inequalities and we take:

$$\prod_1^n (S - x_i) \geq (n-1)^n \prod_1^n x_i$$

or,

$$\frac{\prod_1^n (S - x_i)}{(n-1)^n S^n} \geq \frac{\prod_1^n x_i}{S^n}$$

$$\frac{\prod_1^n (S - x_i)}{\Big[\sum_1^n (S - x_i)\Big]^n} \geq \frac{\prod_1^n x_i}{S^n}$$

An application of (19) is the next. We take:

$$x_1 = -(n-2)a_1 + a_2 + ...a_n > 0$$

$$x_2 = a_1 - (n-2)a_2 + ...a_n > 0$$

$$..... \qquad ....$$

$$x_n = a_1 + a_2 + ...(n_2)a_n > 0$$

from (19) follows:

$$a_1 a_2 ....a_n \geq \prod_1^n [-(n-2)a_1 + a_2 + ...a_n]$$

the well known inequality.

The majorization method
We will prove the inequality (17) as an example of the majorization method.
We will prove:

$$\Big[\frac{(a_1+a_2)}{2}\frac{(a_2+a_3)}{2}\frac{(a_3+a_1)}{2}\Big]^2 \geq a_1 a_2 a_3 \Big[\frac{a_1+a_2+a_3}{3}\Big]^3$$

For $a_1 \leq a_2 \leq a_3$ positive numbers.
Let it be

$$x_1 = a_1,\ x_2 = a_2,\ x_3 = x_4 = x_5 = \frac{x_1+x_2+x_3}{3},\ x_6 = a_3$$

$$y_1 = y_2 = \frac{a_1+a_2}{2},\ y_3 = y_4 = \frac{a_1+a_3}{2},\ y_5 = y_6 = \frac{a_2+a_3}{2}$$

We suppose that

$$x_1 \leq x_{23} = x_4 = x_5 \leq x_6$$

We also see that

$$y_1 = y_2 \leq y_3 = y_4 \leq y_5 = y_6$$

It is no difficult to se that
$x_1 \leq y_1$
$x_1 + x_2 \leq y_1 + y_2$
....
x$_1$ + ... + $x_5 \leq y_1 + ...y_5$

$x_1 + ... + x_6 = y_1 + ... + y_6$
Therefore, taking in mind, that Lnx is concave, according the Majorization theorem (17) true.
(The case $x_1 \leq \frac{x_1+x_2+x_3}{3} 3 \leq x_2 \leq x_3$ is similar).
Another example for a Geometric inequality.
Let A, B, C be the angles of the triangle ABC. We suppose that $A \leq B \leq C$. Obviously,

$$A \leq \pi/3, \quad A + B \leq \pi/3 + \pi/3, \quad A + B + C = \pi$$

Therefore, $(A, B, C) \prec (\pi/3, \pi/3, \pi/3)$, also $(0, 0, \pi) \prec (A, B, C)$.
The function $y = \sin x$ is concave, hence:

$$0 < \sin A + \sin B + sinC \leq 3 \sin \pi/3 = 3\sqrt{3}/2$$

The function $y = Ln \sin x$ is concave, hence:

$$0 < \sin A \sin B \sin C \leq (\sin \pi/3)^3.$$

Remark.
Using the solution of the problem 1, we can prove the conjecture (for a simplex) of [1] 7.14 page 338.

$$\sum_{1}^{n+1} \frac{A_i A_i''}{A_i' A_i''} \geq n^2 - 1.$$

Indeed, we have:

$$\frac{MA_i''}{A_i' A_i''} = m_i, \ (a) \qquad \frac{MA_i'}{A_i A_i'} = q_i, \ (b)$$

from (b) follows that:

$$\frac{A_i' A_i'' - MA_i''}{A_i A_i'' + A_i' A_i''} = q_i \Rightarrow$$

$$\frac{1 - \frac{MA_i''}{A_i' A_i''}}{1 + \frac{A_i A_i''}{A_i' A_i''}} = q_i \Rightarrow$$

$$\frac{1-m_i}{1+\frac{A_iA_i''}{A_i'A_i''}} = q_i \Rightarrow$$

$$\frac{A_iA_i''}{A_i'A_i''} = \frac{1-m_i-q_i}{q_i}$$

but we proved in problem 1 that $m_i = \frac{1-q_i}{n}$, therefore

$$\frac{A_iA_i''}{A_i'A_i''} = \frac{(n-1)(1-q_i)}{nq_i} = \frac{(n-1)}{n}\Big[\frac{q_1+q_2+..q_{i-1}+q_{i+1}..q_{n+1}}{q_i}\Big]$$

and finally,

$$\sum_1^{n+1}\frac{A_iA_i''}{A_i'A_i''} = \frac{n-1}{n}\sum_{i\geq j}^{1,n+1}\Big[\frac{q_i}{q_j}+\frac{q_j}{q_i}\Big] \geq n^2-1.$$

The above in [1] is for a triangle.

References

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