An inequality in the Cartesian plane

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Problem

Let xOy the Cartesian plane and the four points A(1,0), B(-1,0), C(0,1), D(0,-1). Prove that for every point P, so that $OP \geq 1$ it holds:

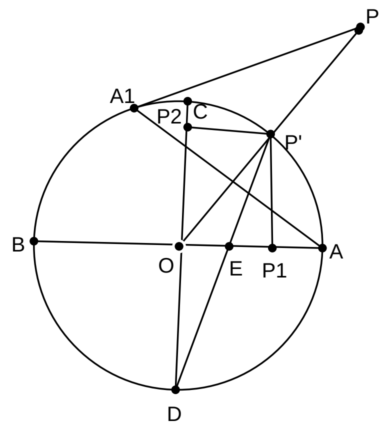
$$|PA - PB| + |PC - PD| \geq 2$$

where $|\vec{MN}|$ is denoted by MN.

Proof.

We consider the circle g=(O,1) and A(1,0), B(-1,0), C(0,1), D(0-1) the intersections on the Ox and Oy axes and x_1, y_1 the Cartesian coordinates of the point $P' = OP \cap g$.
We drow the perpendiculars $P'P_1$ to AB and $P'P_2$ to CD. Let E be the intersection of the lines $P'D$ and AB. The triangles DOE and $P'P_2C$ are similar, hence it follows:

$$OD \cdot CP_2 = OE \cdot P'P_2 = OE \cdot OP_1. \tag{1}$$



From the triangle $AP'B$ follows:

$$BP'^2 - AP'^2 = 2AB \cdot OP_1 \tag{2}$$

$$\frac{BP'}{AP'} = \frac{BE}{AE} \tag{3}$$

From (3) we take

$$\frac{BP'}{AP'} = \frac{1+OE}{1-OE}$$

and then

$$OE = \frac{BP' - AP'}{AP' + BP'} \tag{4}$$

From (1), (2) and (4) we easily take:

$$OE \cdot OP_1 = \frac{(BP' - AP')^2}{4} \tag{5}$$

From (1),(5) follows:

$$1 - x_1 = \frac{(BP' - AP')^2}{4}$$

and then

$$|BP' - AP'| = 2\sqrt{1 - x_1}$$

and similarly

$$|CP' - DP'| = 2\sqrt{1 - y_1}$$

and by the equivalence

$$(1 - x_1)(1 - y_1) \geq 0 \quad \Leftrightarrow \sqrt{1 - x_1} + \sqrt{1 - y_1} \geq 1$$

follows that

$$|P'A - P'B| + |P'C - P'D| \geq 2. \tag{6}$$

Suppose now that $BP > AP$. We take a the point A_1 symmetric of the point A relative to the line OP'. The quadrilateral PA_1BA is trapezion therefore, it is very simple to see that: $PB + PA_1 > P'B + PA_1$ or $PB + P'A > PA + P'B$ and finally

$$PB - PA > P'B - P'A$$

Similarly

$$PC - PD > P'C - P'D$$

therefore, from (6) we take:

$$|PA - PB| + |PC - PD| \geq 2$$