The applications of Leibniz's formula in Geometry.

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Introduction

It is well known from Mechanics, that the polar moment of inertia of a system of weighted points is minimum about the centroid. This can be expressed geometrically and very interesting results can be obtained. In fact, the following formula is known.

Let S be a set of points $A_1, A_2, ...A_n$ in E^d with masses $m_1, m_2, ...m_n$ respectively and G their centroid. For every point P holds:

$$m\Big[\sum_{i=1}^{n} m_i|\vec{PA_i}|^2\Big] = m^2|\vec{PQ}|^2 + \sum_{i\geq j}^{1,n} m_i m_j |\vec{A_iA_j}|^2, \quad (1)$$

where Q is the centroid defined by:

$$\vec{OG} = \frac{\sum_i^n m_i \vec{OA_i}}{m}, \quad m = \sum_{i=1}^{n} m_i \neq 0, \quad m_i \in R.$$

Formula (1) was known to Lagrange, but it seems that first has been used by Leibniz, so we call it, the formula of Leibniz..

The geometric significance of the formula (1) is connected with the use of the barycentric coordinates. Indeed, suppose that $A_1, A_2, ..A_n$ be a (n-1)simplex in $E^{(n-1)}$ and $(m_1/m, m_2/m, ..m_n/m)$ the barycentric coordinates of the point Q, that is

$$\vec{OG} = \frac{\sum_i^n m_i \vec{OA_i}}{m}, \quad m = \sum_{i=1}^{n} m_i \neq 0, \quad m_i \in R.$$

The point Q can be considered as the centroid of the points $A_1, A_2, ..A_n$ with masses $m_1/m, m_2/m, ..m_n/m$ and is easily understood that (1) has geometric significance.

From the formula (1) we can take the very interesting geometric inequality, below:

$$m\Big[\sum_{i=1}^{n} m_i|P\vec{A}_i|^2\Big] \geq \sum_{i\geq j}^{1,n} m_i m_j |A_i\vec{A}_j|^2, \tag{2}$$

The equality holds, if and only if $P = Q$.
Several Mathematicians have used the inequality (2) in proving geometrc inequalities, however, I think, that nobody has deeply investegated the large number of possibilities of the Leibniz's formula. In this paper we will try to discribe only a small part of the various and strong possibilities of this formula.
The paper is divided into four parts. In the first part we give the proof of the Leibniz's formula and a generalization. In the second part we will use it for the proof of some geometric theorems. In the third part we give some interesting quadritic forms and the last part, includes a large number of geometric inequalities.

1. Proof of the Leibniz's fopmula.

It is easily understood that

$$mP\vec{Q} = \sum_{i=1}^{n} m_i P\vec{A}_i$$

$$or \quad P\vec{Q}[mP\vec{Q} - \sum_{i=1}^{n} m_i P\vec{A}_i] = 0$$

$$or, \;\; mP\vec{Q}^2 = P\vec{Q}\sum_{i=1}^{n} m_i P\vec{A}_i + P\vec{Q}\sum_{i=1}^{n} m_i Q\vec{A}_i, \;\; because \;\; of \quad \sum_{i=1}^{n} m_i Q\vec{A}_i = \vec{0}$$

$$that \;\; is \quad mP\vec{Q}^2 = \sum_{i-1}^{n} m_i(P\vec{A}_i + Q\vec{A}_i)P\vec{Q}$$

$$or \;\; mP\vec{Q}^2 = \sum_{i=1}^{n} m_i(P\vec{A}_i + Q\vec{A}_i)(P\vec{A}_i - Q\vec{A}_i)$$

$$or \quad mP\vec{Q}^2 = \sum_{i=1}^{n} m_i P\vec{A}_i^{\,2} - \sum_{i=1}^{n} m_i Q\vec{A}_i^{\,2}$$

Or,

$$\sum_{i=1}^{n} m_i P\vec{A}_i^{\,2} = mP\vec{Q}^2 + \sum_{i=1}^{n} m_i Q\vec{A}_i^{\,2}. \tag{3}$$

Several authors are reffered to (3) as the formula of Leibniz. From (3) we can easily take (1).
Indeed

$$\sum_{i,j}^{1,n} m_i m_j \vec{A_i A_j}^2 = \sum_{i,j}^{1,n} m_i m_j (\vec{OA_i}^2 + \vec{OA_j}^2) - 2\vec{OA_i}\vec{OA_j} =$$

$$= 2m \sum_{i=1}^{n} m_i \vec{OA_i}^2 - 2\sum_{i,j}^{1,n} m_i m_j \vec{OA_i}\vec{OA_j}$$

Or

$$\sum_{i,j}^{1,n} m_i m_j \vec{A_i A_j}^2 = 2m \sum_{i=1}^{n} m_i \vec{OA_i}^2 - 2[\sum_{i=1}^{n} \vec{OA_i}]^2. \tag{4}$$

Formula (4) for $O = Q$ gives:

$$\sum_{i \geq j}^{1,n} m_i m_j \vec{A_i A_j}^2 = m \sum_{i=1}^{n} m_i \vec{QA_i}^2 \tag{5}$$

Consequently from (3) and (5) we have Leibniz's formula (1), see [6].

An extension of Leibniz's formula

We consider a second point P'. The formula (1) can be extended as follows¿

$$m\Big[\sum_{i=1}^{n} m_i \vec{PA_i} \cdot \vec{PA_j}\Big] = m^2 \vec{PG} \cdot \vec{P'G} + \sum_{i \geq j}^{1,n} m_i m_j \vec{A_i A_j}^2 \tag{6}$$

Proof

For every point P, P' we have:

$$m \sum_{i=1}^{n} m_i \vec{PA_i}^2 = m^2 \vec{PQ}^2 + \sum_{i \geq j}^{1,n} m_i m_j \vec{A_i A_j}^2$$

$$m \sum_{i=1}^{n} m_i \vec{P'A_i}^2 = m^2 \vec{P'Q}^2 \sum_{i \geq j}^{1,n} m_i m_j \vec{A_i A_j}^2$$

We add up the above inequalities and we substitute

$$\vec{PA_i}^2 + \vec{P'A_i}^2 = (\vec{PA_i} - \vec{P'A_i})^2 + 2\vec{PA_i} \cdot \vec{P'A_i} = \vec{PP'}^2 + 2\vec{PA_i} \cdot \vec{P'A_i}$$

We will have:

$$2m\sum_{i=1}^{n} m_i \vec{PA_i}\cdot \vec{P'A_i} = m^2(\vec{PG}^2+\vec{PG'}^2-\vec{PP'}^2)+2\sum_{i\geq j}^{1,n} m_i m_j \vec{A_iA_j}^2$$

and finally, taking into account that

$$\vec{PG}^2+\vec{P'G}^2-\vec{PP'}^2 = 2\vec{PG}\cdot\vec{P'G},$$

we have (5).
It is worthwhile to point out here that for a smplex $s^{n-1} = A_1A_2...A_n$ and a point Q with barycentric coordinates $m_1, m_2, ..m_n$, $\sum_{i=1}^{n} m_i = 1$ and two points P, P' *so that* $\angle PQP' \leq \pi/2$, holds:

$$\sum_{i=1}^{n} m_i|\vec{PA}||\vec{P'A_i}| \geq \sum_{i\geq j}^{1,n} m_i m_j |\vec{A_iA_j}|^2 \tag{7}$$

The barycentric coordinates and their geometric significance.
It is useful for the paper to explain clearly the geometric significance of the barycentric coordinates of a point Q with respect to a triangle $A_1A_2A_3$.
Let $m_1,\ m_2,\ m_3$ the barycentric coordinates of a point Q, that is:

$$\vec{OQ} = m_1\vec{OA}_1 + m_2\vec{OA}_2 + m_3\vec{OA}_3, \quad and \quad m_1+m_2+m_3 = 1$$

We suppose that:

$$A_1' = A_1Q\cap A_1A_2,\ \ A_2' = A_2Q\cap A_3A_1,\ \ A_3' = A_3Q\cap A_1A_2$$

and $\overline{F_1},\ \overline{F_2},\ \overline{F_3}$ the oriented areas of the triangles $QA_2A_3,\ QA_3A_1,\ QA_1A_2$.
The formula (7) can be written:

$$\vec{OQ} = m_1\vec{OA}_1 + (m_2+m_3)\frac{m_2\vec{OA}_2+m_3\vec{OA}_3}{m_2+m_3}.$$

From the above follows:

$$\vec{OA_1'} = \frac{m_2\vec{OA}_2+m_3\vec{OA}_3}{m_2+m_3}$$

$$Therefore \quad \frac{\overline{A_2A_1'}}{\overline{A_1'A_3}} = \frac{m_3}{m_2} \tag{8}$$

$$and \;\; from \; (8) \; follows \;\; \frac{m_3}{m_2} = \frac{\overline{F_3}}{\overline{F_2}} \tag{9}$$

Let $\overline{F}$ be the oriented area of the triangle $A_1A_2A_3$. From the above we easily see that $m_1 = \overline{F_1}/\overline{F}$, $m_2 = \overline{F_2}/\overline{F}$, *and* $m_3 = \overline{F_3}/\overline{f}$.
In the n-dimensional Eucledian space, $n+1$ indepentant points, are the vertices of a simplex usually denoted by $s^n = A_1A_2....A_{n-1}$. A point Q with barycentric coordinates $m_1, m_2, ...m_{n+1}$, $m_1 + m_2 + .. \;\; ..m_{n+1} = 1$ is:

$$Q = \sum_{i=1}^{n} m_i A_i, \;\; or \;\; Q = m_1 A_1 + \Big[\sum_{i=2}^{n+1} m_i\Big] \frac{\sum_{i=2}^{n+1} m_i A_i}{\sum_{i=2}^{n+1} m_i} =$$

$$= m_1 + \Big[\sum_{i=2}^{n+1} m_i\Big] Q_1 \;\; hence \;\; \frac{\overline{QQ_1}}{\overline{A_1Q}} = \frac{m_1}{1 - m_1}$$

$$Also \;\; \frac{\overline{QQ_1}}{\overline{A_1Q_1}} = m_1 = \frac{V(QA_2A_3....A_{n+1})}{V(A_1A_2....A_{n+1})}$$

By $V(W)$ is denoted the volume of the body W.
The face opposite the vertex A_i of the n simplex $s^n = A_1A_2....A_{n-1}$ is a $(n-1)$simplex and it is denoted by $s_i^{(n-1)}$. It is useful to point out that the volume of the simplex with vertex Q and opposite face $s_i^{(n-1)}$ is positive if the altitudes from the vertices Q and A_i to the opposite face $s_i^{(n-1)}$ are in the same direction.

2. Propositions from the Geometry.

(a). Euler's formula $IO^2 = R^2 - 2Rr$.
It is easy to see that the incenter I of a triangle ABC has as barycentric coordinates,

$$a/2s, \; b/2s, \; c/2s$$

where a, b, c are the sides of the triangle ABC and $2s = a + b + c$
From the Leibniz's formula follows:

$$2s(aPA^2 + bPB^2 + cPC^2) = 4s^2 \cdot IP^2 + 2sabc$$

or

$$aPA^2 + bPB^2 + cPC^2 = 2s \cdot IP^2 + abc. \tag{10}$$

Now, setting $P = O$ the circumcenter, we take:

$$R^2 \cdot 2s = 2s \cdot IO^2 + abc$$

and easily

$$IO^2 = R^2 - 2Rr$$

Similarly, we can prove the formulas for the excenters $I_a,\ I_b,\ I_c$ that is $I_aO^2 = R^2 + 2Rr_a$ etc., as well as the well known formulas about the segments which join the centroid G the orthocenter H, the incenter I and the circumcenter O, like

$$OH^2 = 9R^2 - \sum a^2$$

or

$$18s \cdot GI^2 = 4s(ab + bc + ca) - \sum a^3 - 15abc$$

and from those,follow very interesting inequalities, like the inequality below

$$4s(ab + bc + ca) \geq \sum a^3 + 15abc \tag{11}$$

(b). The area of the pedal triangle.
Let Q be a point of the plane of the triangle ABC and $Q_1,\ Q_2,\ Q_3$ its projections and $q_1,\ q_2,\ q_3$ its distances from the sides respectively. Denoting by $(\overline{LMN})$ the oriented area, we have:

$$\frac{(\overline{Q_2QQ_3})}{(\overline{ABC})} = \frac{q_2q_3}{bc}$$

and easily follows:

$$\overline{(Q_1Q_2Q_3)} = \frac{\overline{(ABC)}}{4R^2}[m_2m_3a^2 + m_3m_1b^2 + m_1m_2c^2]$$

where $m_1,\ m_2,\ m_3$ are the barycentric coordonates of the point Q.
Formula (1) for $P = O$ (the circumcenter) can be written

$$R^2 = OQ^2 + m_2m_3a^2 + m_3m_1b^2 + m_1m_2c^2$$

The above formulas give the well known formula for the area of the pedal triangle of the point Q.

$$\frac{(Q_1Q_2Q_3)}{(ABC)} = \frac{|\Delta(Q)|}{4R^2} \tag{12}$$

where by $\Delta(Q)$ we denote the power of the point Q.

Quadritic forms via the Leibniz's formula.

Suppose that $s^{(n)} = A_0A_1A_2....A_n$ be a n-simplex in E^n. Formula (2) for $P = A_0$ can be written.

$$m\sum_{i=1}^{n} m_i \vec{A_0A_i}^2 \geq \sum_{i\geq j}^{1,n} m_i m_j \vec{A_iA_j}^2 \tag{13}$$

we set, $m_i = x_i$, $|\vec{A_iA_j}| = a_{ij} = a_{ji}$.
From (10), we take the no negative quadritic form Φ_0,

$$\Phi_0 = \sum_{i=1}^{n} a_{0i}^2 x_i^2 + \sum_{i\geq j}^{1,n}(a_{0i}^2 + a_{0j}^2 - a_{ij}^2)x_i x_j \geq 0. \tag{14}$$

From formula (1), setting $|\vec{PA_i}| = R_i$, we take the no negative quadritic forme Φ.

$$\Phi = \sum_{i=1}^{n} R_i^2 x_i^2 + \sum_{i\geq j}^{1,n}(R_i^2 + R_j^2 - a_{IJ}^2)x_i x_j \geq 0. \tag{15}$$

We assume now that $m = \sum_{i=1}^{n} m_i = 0$, From the formula (4) we take the remarkable no positive quadritic form Φ_1.

$$\Phi_1 = \sum_{i,j}^{1,n} m_i m_j a_{ij}^2 \leq 0. \tag{16}$$

Formulae (14),(15) and (16) give interesting inequalities even for the two dimensional case. We will give some examples working with the formula (16).
Let $m_1 = x_2 - x_3$, $m_2 = x_3 - x_1$, $m_3 = x_1 = x_2$, $x_i > 0$.
From (16) we have:

$$\sum (x_1 - x_2)(x_! - x_2)a^2 \geq 0 \quad (Schur's\ \ inequality). \tag{17}$$

Also from (16) for $m_1 = x_1x_2 - x_1x_3$ (cyclicaly), we take.

$$\sum a^2 x_2^2 x_3^2 \geq x_1x_2x_3 \Big[\sum a(x_2 + x_3 - x_1) \Big]. \tag{18}$$

From (17) we obtain

$$\sum x_1^2 \geq 2x_2x_3\ cosA. \tag{19}$$

The inequality (19) leads us to very interesting results, e.g. setting $x_1 = 1/a,\ \ x_2 = 1/b,\ \ x_3 = 1/c$ we have:

$$\sum a^5 + 2abc \cdot s \geq \sum a^3(b+c) \tag{20}$$

Now for $x_1 = a,\ \ x_2 = b,\ \ x_3 = c$ we take:

$$\sum a^3 + 3abc \geq \sum a^2(b+c) \tag{21}$$

for $x_1 \sim ax_1,\ \ x_2 \sim bx_2\ \ and\ \ x_3 \sim cx_3$, take

$$x_1x_2x_3\Big[\sum a^3x_1\Big] + abc\Big[\sum x_1^2x_2^2\Big] \geq x_1x_2x_3\Big[\sum a^2(bx_2+cx_3)\Big] \tag{22}$$

From the formula (2), for the triangle ABC we have

$$m\sum m_1PA^2 \geq \sum m_2m_3a^2 \tag{23}$$

We set $P = O$, the circumcenter, and $m_i = x_i$. Easily we take:

$$\sum x_i^2 + 2\sum x_1x_2 \cos 2A \geq 0 \tag{24}$$

The well known quadritic form, see[3].

4. Geometric inequalities.

4.1 For the triangle ABC and for $P = O$ (the circumcenter) formula (2) can be written:

$$R^2m^2 \geq \sum m_2m_3a^2. \tag{25}$$

The equality holds for $Q = O$ where Q is the point with barycentic coordinates $m_1/m,\ \ \ m_2/m,\ \ \ m_3/m$. That is the equality holds, if and only if

$$\frac{(BOC)}{m_1} = \frac{(COA)}{m_2} = \frac{(AOB)}{m_3}$$

or, after some calculations

$$a^2(b^2+c^2-c^2)/m_1 = b^2(a^2+c^2-b^2)/m_2 = c^2(a^2+b^2-c_2)/m_3 \tag{26}$$

Neuberg-Pedoe's inequality.
Let ABC, $A'B'C'$ be two triangles, F, F' their areas, and

$$P = \sum a^2(a'^2 + b'^2 - a'^2)$$

We will show the well known Neuberg-Pedoe's inequality, that is:

$$P \geq 16F \cdot F' \tag{27}$$

Indeed, we set $m_! = a^2(b'^2 + c'^2 - a'^2)$, cyclicaly for m_2, m_3 in (25) and immediately follows the Neuberg-Pedoe's inequality. The equality from (26).

Remark. From this point and for the next inequalities the equality case is left for the reader, except, possibly for some remarkable problems.
For $m_1 = a$, $m_2 = b$, $m_3 = c$, from (25), we take:

$$a + b + c \geq 4F/R \tag{28}$$

For $m_1 = a^2$, $m_2 + b^2$, $m_3 = c^2$ from (25) we take:

$$a^2 + b^2 + c^2 \geq 4F\sqrt{3} \tag{29}$$

For $m_1 - b^2c^2$, $m_2 = c^2a^2$, $m_3 = a^2b^2$ and always from (25), we take:

$$\sum b^2c^2 \geq 16F^2 \tag{30}$$

For $m_1 = a^4$, $m_2 = b^4$, $m_3 = c^2$ it follows that:

$$a^4 + b^4 + c^4 \geq 16F^2 \tag{31}$$

For $m_1 = m_2 = m_3 = 1$ it follows the well known inequality

$$9R^2 \geq a^2 + b^2 + c^2 \tag{32}$$

For two triangles ABC, $A'B'C'$ and for $m_1 = a^2a'^2$, $m_2 = b^2b'^2$, $m_3 = c^2c'^2$, we have:

$$\sum a^2a'^2 \geq 16FF' \tag{33}$$

We continue with the two triangles.
For $m_1 = a^2/r'_br'_c$ symmetrically m_2, m_3 and r_a, r_b, r_c, r'_a, r'_b, r'_c the exradii. We take:

$$\sum \frac{a^2}{r'_br'_c} \geq \frac{4F}{F'} \tag{34}$$

For $m_1 = a^2 r'_a$ and symmetrically for the others m

$$\sum a^2 r'_a \geq 4Fs' \tag{35}$$

For $m_1 = a/a'$ etc.

$$\sum \frac{a}{a'} \geq \frac{23^{1/4}}{\sqrt{RR'}} \frac{F^{3/4}}{F'^{1/4}} \tag{36}$$

For $m_1 = \frac{a'}{b'c'}$ etc.

$$\frac{9RR'}{4F'} \geq \frac{1}{\sqrt{3}} \left[\sum a/a' \right] \tag{37}$$

For

$$m_1 = \frac{a^2 b' c'}{(a' + b')(a' + c')},$$

symmetrically for $m_2, \; m_3$

$$\sum a^2 b' c' (b' + c') \geq 2^5 3^{1/2} F F' R' \tag{38}$$

For $m_1 = ax, \;\; m_2 = by, \;\; m_3 = cz$ it follows that:

$$\left[\frac{ax + by + cz}{4F} \right]^2 \geq \sum \frac{yz}{bc} \tag{39}$$

For $m_1 = a^2 x, \;\; m_2 = b^2 y, \;\; m_3 = c^2 z$, we take:

$$\left[\frac{a^2 x + b^2 y + c^2 z}{4F} \right] \geq xy + yz + zx \tag{40}$$

For $m_1 = x, \;\; m_2 = y, \;\; m_3 = z$ we have:

$$R^2 (x + y + z)^2 \geq xyc^2 + yza^2 + zxb^2 \tag{41}$$

M. Klamkin proved (39),(40),(41) in [3]. For $m_1 = \frac{1}{bc}$ cyclically for $m_2, \; m_3$ we have:

$$\sum \frac{1}{ab} \geq \frac{1}{R^2} \tag{42}$$

We suppose that $m_1 + m_2 + m_3 = 0$, from (16) we have:

$$\sum m_2 m_3 a^2 \leq 0$$

let now $m_1 = (b-c)/r_a$ cyclically for $m_2,\ \ m_3$. It is easy to see $\sum \frac{b-c}{r_a} = 0$. consequently, it follows

$$\sum \frac{(a-b)(a-c)}{r_b r_c} a^2 \geq 0 \tag{43}$$

For $m_1 = r_a/bc$, cyclically for $m_2,\ m_3$. We take:

$$\left[\frac{R}{r} - \frac{1}{2}\right]^2 \geq \sum \frac{r_b r_c}{bc} \tag{44}$$

For $m_1 = r_b + r_c,\ \ m_2 = r_c + r_a,\ \ m_3 = r_a + r_b$, we will have:

$$\frac{R(r+4R)^2}{2(2R-r)} \geq s^2 \tag{45}$$

this is a new r, R, s inequality. For $m_1 = (r_b + r_c)/r_a$, cyclically for $m_2,\ m_3$, it follows that:

$$s^2 \geq 10R^2 + \frac{3}{2}Rr - 2\frac{R^3}{r}. \tag{46}$$

Another new r, R, s inequality.

4.2 The inequalities which follow are refered to the distances of a point P from the vertices of the triangle ABC
The formula (2) for the triangle ABC can be written.

$$m \sum_1 R_1^2 \geq \sum m_2 m_3 a^2 \tag{47}$$

where $R_1 = AP,\ \ R_2 = BP,\ \ R_3 = CP$, and $m_1/m,\ m_2/m,\ m_3/m$ the barycentri coordinates of the point Q.

For $m_1 = a,\ m_2 = b,\ m_3 = c$ we take.

$$aR_1^2 + bR_2^2 + cR_3^2 \geq abc \tag{48}$$

a well known inequality, see [4].
The equality holds if and only if P coincides with the point I (the incenter).
M.S. Klamkin proved the inequality

$$aR_2R_3 + bR_3R_1 + cR_3R_1 \geq abc \tag{49}$$

using an inversion (P, k^2) see [4].
It is possible to obtain (49) directly from (47), setting $m_1 = a/R_1,\ \ m_2 = b/R_2,\ \ m_3 = c/R_3$, or $m_1 = aR_2R_3$ etc.
Another nice inequality can be produced from (49) the following.

$$\frac{R_1}{a} + \frac{R_2}{b} + \frac{R_3}{c} \geq \sqrt{3}$$

A remarkable notice here is that the three last inequalities can be proved using identities of the complex plane eg. for (49)

$$\sum (z_1 - z_2)(z - z_1)(z - z_2) = (z_1 - z_2)(z_2 - z_3)(z_3 - z_1)$$

etc.
We denote by $l_1, l_@, l_3$ the barycentric coordinates of the point P and by ρ_1, ρ_2, ρ_3 the circumradii of the triangles $BPC,\ CPA,\ APB$, (49) can be written.

$$l_1\rho_1 + l_2\rho_2 + l_3\rho_3 \geq R \qquad (49a)$$

For $m_1 = a^2,\ \ m_2 = b^2,\ \ m_3 = c^2$, we will have:

$$\sum (b + c)R_1^2 \geq (a + b)(b + c)(c + a)/4 \tag{50}$$

For $m_1 = b^2c^2$ cyclically for the others m, we take:

$$\sum b^2c^2R_!^2 \geq a^2b^2c^2 \tag{51}$$

For $m_1 = a(b + c)$ cyclically the $m_2,\ m_3$, we take:

$$\sum a(b + c)R_1^2 \geq \frac{2abc}{3}(a + b + c) \tag{52}$$

For $m_1 = bc,\ m_2 = ca,\ m_3 = ab$, it follows that:

$$\sum bcR_1^2 \geq abc \cdot 2s/3 \tag{53}$$

For $m_1 = b^2 + c^2$ cyclically the m_2, m_3 we have:

$$\sum (b^2 + c^2)R_1^2 \geq \frac{2}{3}(a^2b^2 + b^2c^2 + c^2a^2) \tag{54}$$

For $m_1 = \frac{bc}{b+c}$ cyclically the $m_2,\ m_3$. We take:

$$\sum \frac{bc}{b + c}R_1^2 \geq \frac{2abc}{(a + b)(b + c)(c + a)} \tag{55}$$

For $m_1 = a^2/bc,\ \ m_2 = b^2/ca,\ \ m_3 = c^2/ab$, we take:

$$\sum \frac{a^2}{bc} R_!^2 \geq \frac{abc(ab+bc+ca)}{a^3+b^3+c^3} \tag{56}$$

For $m_1 = r_!^2/a^2,\ \ M_2 = r_2^2/b^2,\ \ M_3 = r_3^2/C^2$ we will have

$$\sum \frac{R_!^2}{a^2} \geq \frac{a^4+b^4+c^4}{a^2b^2+b^2c^2+c^2a^2} \geq 1 \tag{57}$$

For $m_1 = \frac{1}{b^2+c^2}$ cyclically for m_2, m_3. We will take:

$$\sum \frac{R_1^2}{b^2+c^2} \geq \frac{1}{2.5} \tag{58}$$

For $m_1 = [\frac{r_b r_c}{r_a}]^2$ and cyclically for $m_2,\ m_3$, we will have:

$$\sum R_!^2 \cot \frac{A}{2} \geq 2F \tag{59}$$

We will continue. We will give only the m_1 and $m_2,\ m_3$ will follow cyclically a cause of symmetry.
-For $m_1 = r_a$

$$\sum r_a R_1^2 \geq \frac{12F^2}{r+4R} \tag{60}$$

-For $m_1 = \frac{bc}{R_1}$

$$\Big[\sum bc\Big]\Big[\sum \frac{bc}{R^2}\Big] \geq \Big[\sum \frac{a^3}{R_2^2 R_3^2}\Big] abc \tag{61}$$

-For $m_1 = 1/R_1$

$$(\sum R_1)(\sum R_2R_3) \geq \sum a^2 R_1 \ \ and \ \ (\sum R_1)^2(\sum R_2R_3) \geq 16F^2 \tag{62}$$

-For $m_1 = p_2p_3$ where $p_1,\ p_2,\ p_3$ are the distances of the point P from their sides.

$$\Big[\sum p_1\Big]\Big[\sum \frac{1}{p_2}\Big] \geq \frac{4F^2}{\sum p_2p_3R_1^2} \tag{63}$$

-For $m_1 = \frac{1}{R_2R_3}$

$$\Big[\sum R_1^2\Big]\Big[\sum R_1\Big] \geq \sum a^2 R_2R_3 \tag{64}$$

-For $m_1 = \frac{1}{R_1^2}$

$$\sum R_1^2 R_2^2 \geq \frac{16}{9} F^2 \tag{65}$$

-For $m_1 = \frac{bc}{R_1}$

$$(\sum bcR_1)(\sum bcR_2R_3) \geq abc\frac{64F^3}{(\sum R_1)^3} \tag{66}$$

-For $m_1 = \frac{bc}{R)2R_3}$

$$(\sum bcR_1^3)(\sum aR_1) \geq abc(\sum a^3 R_2 R_3) \tag{67}$$

-For $m_1 = \frac{a}{R_2R_3}$

$$(\sum aR_1^2)(\sum aR_1) \geq a^2b^2c^2 \tag{68}$$

-For $m_1 = R_1$

$$(\sum R_1)(\sum R_1^2)(\sum R_1^3) \geq a^2b^2c^2 \tag{69}$$

-For $m_1 = \frac{a}{R_1}$

$$\Big[\sum aR_1\Big]\Big[\sum \frac{a}{R_1}\Big] \geq \frac{a^2b^2c^2}{R_1^2R_2^2R_3^2} \tag{70}$$

-For $m_1 = \sqrt{\frac{ap_2p_3}{bcp_1}}$

$$\Big[\sum \frac{a}{p_1}\Big]\Big[\sum \frac{a}{p_1}\Big] \geq \frac{8F^2R}{p_1p_2p_3} \tag{71}$$

For the two triangles and a point P for $m_1 = a',\ m_2 = b',\ m_3 = c'$ we have

$$\sum a'R_1^2 \geq \frac{\sum b'c'a^2}{a' + b' + c'} \tag{72}$$

-For $m_1 = b'c'$

$$\sum b'c'R_1^2 \geq \frac{a'b'c'(\sum a'a^2)}{a'b' + b'c' + c'a'} \tag{73}$$

4.3 Inequalities for a $(n-1)$-simplex.

Let $s^{(n-1)} = A_1A_2....A_n$ be a $(n-1)$-simplex in $E^{(n-2)}$ and Q a point with barycentrc coordinates $m_1, m_2, m_3, ...m_n$, that is $\sum_{i=1}^{n} m_i = 1$. Leibniz's formula for a point can be written.

$$\sum_{i=1} n m_i R_i^2 = \vec{PQ}^2 + \sum_{i \geq j}^{1,n} m_i m_j a_{ij}^2 \tag{74}$$

where $|\vec{A_iP}| = R_i$ and $|\vec{A_iA_j}| = a_{ij}$.
We take $P = O$ the circumcenter. From (71), it follows that:

$$R^2 = \vec{OQ}^2 + \sum_{i \geq j}^{1,n} m_i m_j a_{ij}^2.$$

Therefore, denoting by $\Delta(Q)$ the power of the point Q, we have:

$$\Delta(Q) = \sum_{i \geq j}^{1,n} m_i m_j a_{ij}^2 \tag{75}$$

and easily we take:

$$R^2 \geq \sum_{i \geq 1}^{1,n} m_i m_j a_{ij}^2. \tag{76}$$

The equality for $Q = O$.
Let now $P = Q$. From (74) we have

$$\sum_{i=1}^{n} m_i R_i^2 = \sum_{i > j}^{1,n} m_i m_j a_{ij}^2 \tag{77}$$

consequently by (75),(77) it follows that:

$$\Delta(Q) = \sum_{i=1}^{n} m_i R_i^2 \tag{78}$$

and from the above and the Gauchy-Schwarz inequality

$$\sum_{i=1}^{n} m_i R_i^2 \geq \frac{(\sum m_i R_i)^2}{\sum m_i}$$

we take:

$$R \geq \sum_{i-1}^{n} m_i R_i \tag{79}$$

From (75) using again the Gauchy-Schwarz's inequality we obtain:

$$R^2 \geq \sum_{i>j}^{1,n} m_i m_j a_{ij}^2 \geq \frac{\left[\sum_{i>j}^{1,n} a_{ij}\right]^2}{\sum_{i>j}^{1,n} m_i m_j} \qquad (a)$$

or,

$$R^2 \Big[\sum_{i>j}^{1,n} m_i m_j\Big] \geq \Big[\sum_{i>j}^{1,n} m_i m_j\Big]^2 \qquad (b)$$

'An interesting inequality can be produced from the above.

$$1 = [\sum_{i=1}^{n} m_i]^2 = \sum_{i=1}^{n} m_i^2 + 2\sum_{i\geq j}^{1,n} m_i m_j$$

or

$$1 \geq \frac{2}{n-1} \sum_{i\geq j}^{1.n} m_i m_j + 2\sum_{i\geq j}^{1.n} m_i m_j$$

The last inequality together with the above (b) gives:

$$R\sqrt{\frac{n-1}{2n}} \geq \sum_{i\geq j}^{1,n} m_i m_j a_{ij}. \qquad (80)$$

The equality for (79) holds if and only if $P = O$ (the circumcenter). The equality for (80) just when the simplex is a regular one and Q is its circumcenter.
From the inequalities (79) and (80) we can produce remarkable inequalities for a triangle ABC, e.g. (79) can be written:

$$R \geq m_1 R_1 + m_2 R_2 + m_3 R_3. \qquad (81)$$

Suppose that P is the centroid of ABC. The inequality (81) will give:

$$9R/2 \geq m_a + m_b + m_c. \qquad (82)$$

where m_a, m_b, m_c are the medians of ABC.
If P coincides with the incenter I, the inequality (81) will give:

$$2Rs \geq \sum (b+c) t_a \qquad (83)$$

where t_a is the bissectrice of the angle A.
The inequality (80), for $Q = G$ (the centroid), will give:

$$3R\sqrt{3} \geq a + b + c, \tag{84}$$

the well known inequality.
The inequality (80) for a $(n-1)$simplex and $Q = G$ will give:

$$R \geq \frac{2s}{n^2}\sqrt{\frac{n-1}{2n}} \tag{85}$$

where $2s = \sum_{i>j}^{1,n} a_{ij}$. That is, the maximum sum of the edges, has the regular simplex, from all the inscribed simplices in the $(n-1)$sphere.
Inequalities (49a) and (79) produce very interesting inequalities.
Let P be an interior point in the triangle ABC and $l_1,\ l_2,\ l_3$ its barycentric coordinates with respect to its pedal triangle. We can show:

$$l_1R_1 + l_@R_2 + l_3R_3 \geq 2R_0 \geq 2(l_1p_1 + l_2p_2 + l_3p_3) \tag{86}$$

where R_0 is the circumradius of the pedal triangle $A'B'C'$ of the point P. The inequality (49a) asserts:

$$l_1R_1/2 + l_2R_2/2 + l_3R_3 \geq R_0 \tag{87}$$

consequently, from the above inequalities we can prove (86).
Another interesting application is the proof of the Jung's theorem see[5] on convex sets. That is, a convex set of diameter D in E^n can be in a sphere of radius R so that:

$$R \leq D\left[\frac{n}{2n+2}\right]^{\frac{1}{2}} \tag{88}$$

The proof from the formula (77) for $R_i = R,\ \ a_{ij} \leq D$. Formula (77) for a n-smplex in E^n will be

$$R^2 \leq \sum_{i>j}^{1,n+1} m_im_jD^2$$

but we already found in the page 16 that

$$\sum_{i>j}^{1,n+1} m_im_j \leq \frac{n}{2n+2}$$

and from the above we take (88)
We end the paper keeping the feeling, that Leibniz's formula, has a large number of possibilities and only a few have been investigated here.

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