A classification of a conic through three no colinear given points, according the position of its center. Nine-point ellipse.

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Introduction

We can easily see that there is a conic (g) through three non colinear given points A, B, C, and having its center a fourth given point M. In this note, our first purpose is to classify the conic (g), according the position of the point M and our second one, to generalize the well known nine point circle.

Classification of the conic (g).
Let $x, y, z,\ x+y+z=1$ be the barycentric coordinates of a point $P \in (g)$
The equation of the cinic (g), through the points A, B, C is:

$$(g): \quad f(x,y,z) = pyz + qzx + rxy = 0.$$

The center $M(x_1, y_1, z_1)$ of (g) satisfies the equations:

$$\frac{\vartheta f(x_!, y_1, z_1)}{\vartheta x_1} = \frac{\vartheta f(x_1, y_1, z_1)}{\vartheta y_1} = \frac{\vartheta f(x_1, y_1, z_1}{\vartheta z_1} \tag{1}$$

where $x_1, y_1, z_1, \quad x_1 + y_1 + z_1 = 1$ are the barycentric coordinates of the point M, see [1] page 354.
From (1) it follows that:

$$qz_1 + ry_1 = pz_1 + rx_1 = py_! + qx_1. \tag{2}$$

case 1. $x_1(y_1 + z_1 - x_1) \neq 0$.
It follows from (2) that:

$$q = \frac{x_1y_1 + y_1z_1 - y_1^2}{x_1z_1 + x_1y_1 - x_1^2}p, \quad r = \frac{y_1z_1 + z_1x_1 - z_1^2}{x_1z_1 + x_1y_1 - x_1^2}p. \tag{3}$$

We denote

$$U = p^2 + q^2 + r^2 - 2pq - 2qr - 2rp$$

and

$$W = x_1^4 + y_1^4 + z_1^4 - 2(x_1^2 y_1^2 + y_1^2 z_1^2 + z_1^2 x_1^2) = -(1 - 2x_1)(1 - 2y_1(1 - 2z_1)$$

The classification of (g) follows according

$$(a) : U < 0, \quad (b) : U = 0, \quad (c) : U > 0. \tag{4}$$

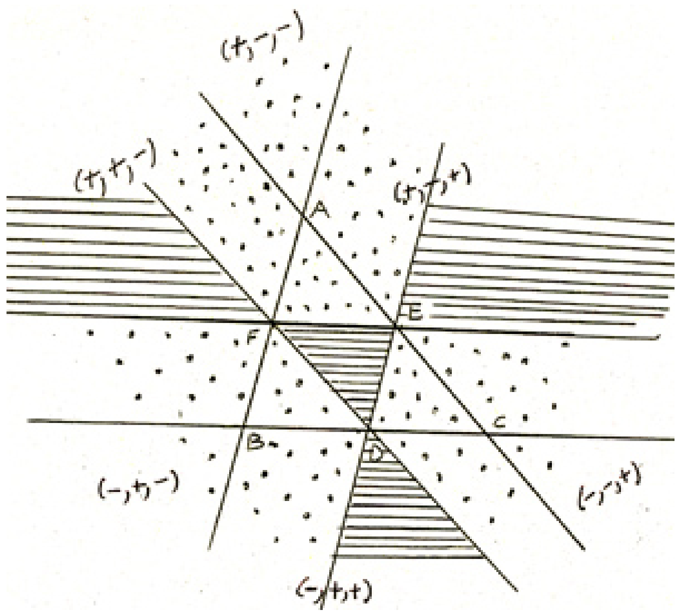
Substitunting p, q, r from (3) to (4) we take equivalently

$$(a) : W < 0, \quad (b) : W = 0, \quad (c) : W > 0. \tag{5}$$

Where (a): means that (g) is an ellipse
(b): means that (g) is a parabola (when M is a point at infinity, otherwise a pair of parallel straight lines.
(c): means that (g) is an hyperbola or a pair of intersecting lines.
See [1] page 358.
Let D, E, F the middlepoints of the sides BC, CA, AB of the triangle ABC, respectively. From (5) we see that (g) is an ellipse, when M lies in the shaded region S_e in the figure, a pair of parallel lines, when M lies in the limited point setDE and an hyperbola in the remaining (doted) plane.
To understand easily the position of the point $M(x_1, y_1, z_1)$, we can consider the areal coordinates

$$x_1 = \frac{(BMC)}{(ABC)}, \quad y_1 = \frac{(CMA)}{(ABC)}, \quad z_1 = \frac{(AMB)}{(ABC)}$$

and their sign as in the figure (inside the triangle ABC, x_1, y_1, z_1 are positive).



Case 2.
(a): $x_1 = 0$. We easily find that $M = D = (0, 1/2, 1/2)$ and

$$(a): \; 4qr > 0 \; (ellipse) \;\; (b): \; 4qr = 0 \; (parabola \; (c): \; 4qr < o \; (hyperbola)$$

That is for the point M in coincidence with the middlepoint of the sides, it is possible for (g) to be an ellipse, parabola or hyperbola.
(b): Suppose that: $y_1 + z_1 - x_1 = 0$ also it is $x_1 + y_1 + z_1 = 1$, so from (2) we find that $p = 0, \; q = r, \; x_1 = 1/2, \; y_1 + z_1 = 1/2,$ also

$$(a): \; 4qr > 0 \; (ellipse) \; (b): \; 4qr = 0 \; parabola \; 4qr < 0 \; (hyperbola)$$

That is for $M \in DE \cup EF \cup FD$ we will have ellipse, parabola or hyperbola according the above.

The nine point-ellipse.
Suppose that $M \in S_e \cup DE \cup EF \cup ED$. Therefore there is an ellipse (g) through the points A, B, C having M as a center. The homothety with center the centroid G of the triangle ABC and ratio -2 transforms M to a point N. We denote by A_1, B_1, C_1 the middle points of the straight line segments NA, NB, NC and by A_2, B_2, C_2 the intersections of the lines AN, BN, CN with BC, CA, AB respectively. We will show that there is an ellipse (w)

passing through the points $D, E, F, A_1, B_!, C_1, A_@, B_2, C_2$ (D, E, F are the middle points of the sides of the triangle ABC). That is, there is a nine point ellipse. It is easily understood that this ellipse (w) coincides with the nine point circle when M is the circumcenter of the triangle ABC,so that N is its orthocenter.

Indeed, we suppose that (g) is the parallel to a direction u projection of a circle (g'), circumscribed to a triangle $A'B'C'$ and every point Q of the plane of the ellipse (g) is the projection of the point Q' of the plane of (g'). It is now obvius ((g') and (w') are homothetic) that the projection of the nine point circle (w') of the triangle $A'B'C'$ is the ellipse (w) through the points $D, E, F, A_1, B_1, C_1, A_2, B_2, C_2$.

. As it is known the parallel projection to a direction u of a plane p' to another plane p is an affine trasformation. So it would be possible to use an affinity to transforme the (g) to $)g')$. We prefered the parallel projection for simplicity.

Refferences

1. C. Smith, Conic sections, Mac Millan and Co, New York 1956, pp 341-366.
2. M. Audin, Geometry, Springer,2002.