Convex figures with conjugate diameters

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Introduction

We define that the diameters PP' and QQ' of the ellipse

$$c : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

are conjugates, if and only if, the tangent at the extremities of PP' QQ' are respectively parallel to QQ' and PP'. The parallelogram also with the sides on the tangents at the points P, P', Q, Q' has constant area equal to $4ab$. So, from the above, arises the question: Is the ellipse the only convex figure with conjugate diameters? In this note we will try to determine the largest class of the convex figures with conjugate diameters.

Definitions,notations, basic concepts.

The diameter of a convex figure K, parallel to the line ϵ, is the maximum chord PP' of K parallel to ϵ. Taking in mind that the support lines at the extremities of a diameter are parallel, the definition of the conjugate diameters for the convex figures will be similar with the definition for the ellipse. That is:

The diameters PP' and QQ' of the convex figure K will be conjugate, if the support lines at the points P, P' and Q, Q' are respectively parallel to the diameters QQ', PP'.

In the following, we shall be concerned with the usual convexity notation, as it can be found in the standard convexity books. The translation of the set K by the vector A is denoted $K + A$ and the homothety relative to the origin O and the ratio λ is $\lambda.K$. The rotation about O and angle ϕ is denoted by: $R(O, \phi)(K)$.

The Minkowski sum of two sets K and F is a new convex set W defined by the relation: $W = [M + N/M \in K, N \in F]$ and is denoted by: $W = K + F$. The symmetroid K^* of the convex figure K is defined by the formula $K^* = \frac{1}{2}[K + (-K)]$.

If PP' is a diameter of K, the points $P_1 = \frac{1}{2}[P - P']$, $P_2 = \frac{1}{2}[P' - P]$ are the extremities of the diameter P_1P_2 of K^*. We easily see that P_1P_2 is parallel and equal to PP'. Also the support lines to K* at the points P_1, P_2 are parallel to the support lines to P, P' of K respectively.
The breadth of K and K^* in every direction is the same and also K and K* have the same perimeter length. Denoting by R the circumradius, r the inradius and A the area of K, it is known that:
a. $R(K^*) \leq R(K)$.
b. $r(K^*) \geq r(K)$.
c. $A(K^*) \geq A(K)$.
See [1],[2].
The solution of the problem.
We first suppose that the convex figure K has smooth boundary, whithout str. line segments and one support line at every point.
We assume that the convex figure K has conjugate diameters. The centrosymmetric figure

$$K^* = \frac{1}{2}[K + (-K)]$$

will have, according Minkowski's addition theorems, conjugate diameters (parallel respectively to the diameters of K) as well.
Let now O be the center of K^* and POP', QOQ' two conjugate diameters. As it is known, there is an affinity transforming the diameters $PP' and QQ'$ to perpendicular equal diameters AOA' and BOB' . The figure K^* is transformed to the centrosymmetric figure F.
Let us suppose that $OA = OA' = OB = OB' = 1$ and S_1, S_2, S_3, S_4 the arcs $AB, BA', A'B', B'A$ of F. Hence the boundary F, denoted by bdF is:

$$bdF = S_1 \cup S_2 \cup S_3 \cup S_4$$

see fig.1.

Let now $M \in S_1$ and ϵ be the support line of S_1 to the point M. The support function of F for the point M is the distance ON' of the point O from ϵ. We denote:

$$ON' = h(S_1, \theta)$$

where θ is the angle of the axis OA with QN'. Let us denote by S_1^p the polar

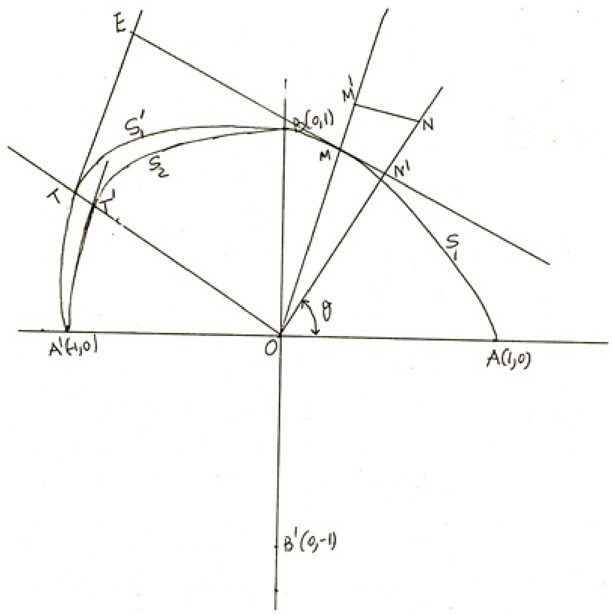

Figure 1:

of S_1 with pole the point O. We will have:

$$ON'.ON = h(S_1, \theta).r(S_1^p, \theta) = 1.$$

We consider the rotation $R(O, \phi)$, and the arc S_1' by:

$$S_1' = R(O, \frac{1}{2}\pi)(S_1^p).$$

The support line of S_1' at the point $T = R(O, \frac{1}{2}\pi)(N)$ is parallel to OM, because it is perpendicular to the support line of S_1^p to the point$N$. (Obviously the support line of S_1^p to the point N is the perpendicular from N to the OM). Hence as we can see from Fig.1 the quadrilateral $OTEM$ is parallelogram. This means that the arcs S_2 and S_1' have for every couple of intersection points T, T', with same radius through O, a parallel tangent. It is a simple problem of the Differential Geometry to prove that these arcs must be parallel, hence the arcs are homothetic. But the arcs have common points, the points B, A', therefore the arcs must be identical. The boundary of F, follows from the next relation.

$$bdF = S_1 \cup R(O, \frac{1}{2}\pi)(S_1^p) \cup S_3 \cup R(O, \frac{1}{2}\pi)(S_3^p). \tag{1}$$

As it known the curve $v = bdF$ has been studied by J. Radon and is called **Radon Curve**.
So we conclude to the following Theorem.

Theorem.
The convex figure K has conjugate diameters if and only if its symmetroid K^* is an affine image of a Radon curve.
We also see from the Fig.1 that:

$$OT.ON' = r(S_1^p, \theta).h(S_1, \theta) = 1.$$

That is the area of the parallelogram $OTEM$ is constant. The affinity transformation preserves the ratio of the area of the regions, therefore the circumscribed parallelograms about K^*, with sides on the support lines on the extremities of two conjugate diameters, will have a constant area. Taking now in mind the properties of K^* relative to the K, we see that the above are valid and for K. The set of the convex figures of the plane with conjugate diameters we will denote by C_v.

Limits.
We can also find some limits of the area of the circumscribed parallelogram E_v of a figure $K_v \in C_v$ We will prove that:

$$\frac{A(K_v)}{A(E_v)} \leq \frac{\pi}{4}. \tag{2}$$

Indeed, if v is the corresponding for the convex figure K_v Radon curve, it is easy to see that its polar dual v^p incloses the same area with v, that is: $A(v^p) = A(v)$. So taking in mind the well known inequality for a convex figure W.

$$A(W).A(W^p) \leq \pi^2,$$

see [3],[4].
we have:

$$A^2(v) \leq \pi^2$$

or

$$A(v) \leq \pi.$$

The parallelogram E_1 with sides on the support lines parallel to the conjugate diameters of v has area 4. Therefore,

$$\frac{A(v)}{A(E_1)} \leq \frac{\pi}{4}.$$

So for K_v^* holds:

$$\frac{A(K_v^*)}{A(E_v^*)} \leq \frac{\pi}{4}$$

and finally

$$\frac{A(K_v)}{A(E_v)} \leq \frac{\pi}{4}.$$

The equality for the ellipse.

References

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