Generalization of Chebyshev inequality.

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The Chebishev's inequality is a very useful tool in investigating problems about inequalities in Algebra, Geometry and Statistics.
Pafnuty Chebyshev(1821-1894), is considered a founding father of the Russian Mathematics. He was professor in st.Petersbourg and is known for his work in the field of Probability, statistics and number Theory. The well known Chebyshev's sum inequality states:
If $A = (a_1, a_2, ...a_k)$ and $X = (x_1, x_2, ...x_k)$ are two real sequences such that:
$a_1 \leq a_2 \leq ..... \leq a_k$ and $x_1 \leq x_2 \leq ...... \leq x_k$ or
$a_1 \geq a_2 \geq ...... \geq a_k$ and $x_1 \geq x_2 \geq ..... \geq x_k$, then
the following inequality is true.

$$\frac{\sum_{i=1}^{k} a_i x_i}{k} \geq \left(\frac{\sum_{i=1}^{k} a_i}{k}\right)\left(\frac{\sum_{i=1}^{k} x_i}{k}\right) \qquad (1)$$

Reversing the inequalities in the sequence of x's the inequality (1) is reversed. The most familiar and practical generalization for m sequences of real numbers is the following, see [1] or [2].

$$A_1 = (a_{11}, a_{12}, ...a_{1k})$$

$$A_2 = (a_{21}, a_{22}, ...a_{2k})$$

....

$$A_m = (a_{m1}, a_{m2}, ...a_{mk})$$

such that $a_{ij} \leq a_{i,j+1}$ for $1 \leq i \leq m$,$1 \leq j \leq k$, or $a_{ij} \geq a_{i,j+1}$,
then the following inequality is true:

$$\frac{1}{k}\sum_{i=1}^{k} a_{1i}a_{2i}.....a_{ki} \geq \left(\frac{\sum_{i=1}^{k} a_{1i}}{k}\right)\left(\frac{\sum_{i=1}^{k} a_{2i}}{k}\right).....\left(\frac{\sum_{i=1}^{k} a_{mi}}{k}\right) \qquad (2)$$

Our generalization has different direction. We have found intermediate terms between the two parts of (1).
Assuming that the terms in (1) are positive we proved:

$$\frac{a_1x_1+..+a_kx_k}{a_1+..+a_k} \geq \frac{1}{k}\left[\frac{a_1x_1+..+a_{k-1}x_{k-1}}{a_1+..+a_{k-1}}+..+\frac{a_2x_2+..+a_kx_k}{a_2+..+a_k}\right] \geq \quad (3)$$

$$\geq \frac{2!}{k(k-1)}\left[\frac{a_1x_1+..+a_{k-2}x_{k-2}}{a_1+..+a_{k-2}}+..+\frac{a_3x_3+..+a_kx_k}{a_3+..+a_k}\right] \geq$$

$$\geq \frac{3!}{k(k-1)(k-2)}\left[\frac{a_1x_1+..+a_{k-3}x_{k-3}}{a_1+..+a_{k-3}}+..+\frac{a_4x_4+..+a_kx_k}{a_4+..+a_k}\right] \geq$$

$$\geq \frac{(t+1)!}{k(k-1)(k-2)..(k-t)}\left[\frac{a_1x_1+..+a_{k-(t+1)}x_{k-(t+1)}}{a_1+..+a_{k-(t+1)}}+...+\frac{a_{t+2}x_{t+2}+.....+a_kx_k}{a_{t+2}+..+a_k}\right] \geq$$

$$.... \geq \frac{x_1+x_2+..+x_k}{k}$$

More formal, our proposition is:
We suppose that $A=(a_1,a_2,....a_k)$ and $X=(x_1,x_2,....x_k)$ are two sequences of positive numbers such that:
$a_1 \leq a_2 \leq .... \leq a_k$ and $x_1 \leq x_2 \leq .... \leq x_k$ or
$a_1 \geq a_2 \geq .... \leq a_k$ and $x_1 \geq x_2 \geq .... \geq x_k$.

We consider $a_1,a_2,....a_k$ as corresponding weights for $x_1,x_2,....x_k$ and we define $f(q)$ as the sum of the weighted means of all the $\binom{k}{q}$ combinations of q of the k values $x_1,x_2,....x_k$. For example:

$$f(k-1)=\frac{a_1x_1+...+a_{k-1}x_{k-1}}{a_1+...+a_{k-1}}+...+\frac{a_2x_2+...+a_kx_k}{a_2+...+a_k}.$$

It is obvious that Chebyshev's inequality (1) can be written as

$$\frac{a_1x_1+...+a_kx_k}{a_1+...+a_k} \geq \frac{x_1+...+x_k}{k} \quad or \quad f(k) \geq \frac{f(1)}{\binom{k}{1}}$$

According to the above formulation our generalization is:
Theorem

$$f(k) \geq \frac{f(q)}{\binom{k}{q}} \geq \frac{f(1)}{\binom{k}{1}} \quad (4)$$

for everyq q, $1 \leq q \leq k$.
The proof of (4) will follow by induction from the inequality

$$f(k) \geq \frac{f(k-1)}{k} \tag{5}$$

hence, assuming that (4) is correcct we will prove that

$$f(k) \geq \frac{f(q-1)}{\binom{k}{q-1}}$$

From (5) for $q \leq k$ we easily see that

$$f(q) \geq (k-q+1)\frac{f(q-1)}{q}$$

therefore

$$f(k) \geq \frac{f(q)}{\binom{k}{q}} \geq \frac{(k-q+1)\frac{f(q-1)}{q}}{\binom{k}{q}} = \frac{f(q-1)}{\frac{k!}{(q-1)!(k-q+1)!}} = \frac{f(q-1)}{\binom{k}{q-1}}$$

We will now prove the inequality (5). For simplicity we put

$$S = a_1x_1 + ... + a_kx_k, \ \ S' = a_1 + ... + a_k$$

so we have to prove

$$\sum_{p=1}^{k} \frac{SS' - Sa_p - (SS' - S'a_px_p)}{S'(S'-a_p)} \geq 0.$$

That is:

$$\sum_{p=1}^{k} \left[\frac{\sum_{i=1}^{k} a_ia_p(x_p - x_i)}{S' - a_p} \right] = \sum_{i>p}^{1,k} a_ia_p(x_p - x_i) \left[\frac{1}{S' - a_p} - \frac{1}{S' - a_i} \right] =$$

$$= \sum_{i>p}^{1,k} \frac{a_ia_p(x_p - x_i)(a_p - a_i)}{(S' - a_p)(S' - a_i)} \geq o$$

The differences $x_p - x_i$ and $a_p - a_i$ are both positive or negative so their product is positive. If the inequalities of the sequence of x's are reversed, then the products are negative, hence the inequalities (3),(5) are reversed.

Applications

Let $a_1, a_2, a_3, ...a_k$ be a sequance of positive numbers and m, n positive integral numbers. From the generaliged Chebyshev's theorem we will have:

$$\frac{a_1^{n+m} + a_2^{n+m} + .. + a_k^{n+m}}{a_1^n + a_2^n + .. + a_k^n} \geq \frac{1}{k}\Big[a_1^m + a_2^m + ..a_k^m\Big] \tag{6}$$

$$\frac{a_1^{n+m} + ..a_k^{n+m}}{a_1^n + ..a_k^n} \geq \frac{(t+1)!}{k(k-1)..(k-t)}\left[\frac{a_1^{n+m} + ..a_{k-(t+1)}^{n+m}}{a_1^n + ..a_{k-(t+1)}^n} + ..\frac{a_{t+2}^{n+m} + ..a_k^{n+m}}{a_{t+2}^n + ..a_k^n}\right] \tag{7}$$

$$\frac{a_1^{n-m} + ..a_k^{n-m}}{a_1^n + ..a_k^n} \leq \frac{1}{k}\Big[a_1^{-m} + ..a_k^{-m}\Big] \tag{8}$$

$$\frac{a_1^{n-m} + ..a_k^{n-m}}{a_1^n + ..a_k^n} \leq \frac{(t+1)!}{k(k-1)..(k-t)}\left[\frac{a_1^{n-m} + ..a_{k-(t+1)}^{n-m}}{a_1^n + ..a_{k-(t+1)}^n} + ..\frac{a_{t+2}^{n-m} + ..a_k^{n-m}}{a_{t+2}^n + ..a_k^n}\right] \tag{9}$$

Remark:For $q + 1 = k - t$ the inequality (4) is transformed to (3).

References

1. G.H.Hardy, J.E.Littlewood, G.Polya, Inequalities, Cambridge University Press 1967.

2. D.S,Mitrinovic Analytic Inequalities Springer-Verlag 1970.