Van den Berg's Theorem

G. A. Tsintsifas
Theassaloniki Greece GR

Let $Q(z) = z^3 + a_1z^2 + a_2z + a_3 = 0$ be a cybic in C and $z_1,\ z_2,\ z_3$ the roots, denoted in the plane by the points A,B,C. The Steiner ellipse in the triangle ABC is denoted by E and $F_1,\ F_2$ the foci. Van den Berg's **theorem** asserts that the roots of the derivate $Q'(z)$ are the complex numbers defined in C by the points F_1 *and* F_2.
I found in my notes the above nice theorem whithout any information about the source. I succed the following interesting proof, but we need a short introduction about the Sreiner ellipse and the affine transformation see(1) and (2).
The Steiner's ellipse E is inscribed in a triangle ABC such that is tangent to the sides in their midpoints. The center of E is the centroid G of the triangle and the midpoints of AG,BG,CG are in E. From the Geometric transformations theory it is well known that an affinity transforms a triangle $A_0B_0C_0$ to another triangle ABC, the parallelism, the ratio of the parallel streight line segments and the ratio of the areas are preserved.
Proof.

Let $A_0B_0C_0$) the equilateral triangle in C so that the vertices $A_0,\ B_0,\ C_0$ are represented by the complex numbers:

$$2,\ \ 2(\cos 120^0 + i\sin 120^0),\ \ 2(\cos 240^0 + i\sin 240^0)$$

respectively.
We consider the affinity T so that:

$$x' = x,\ and\ y' = ky\ \ where\ \ 0 < k \leq 1$$

We can choose $A_0B_0C_0$ *and* k so that the triangle $A_0B_0C_0$ will be trasformed by T to the triangle ABC. Therefore we will have:

$$z_1 = 2$$

$$z_2 = 2(\cos 120^0 + ik \sin 120^0)$$
$$z_3 = 2(\cos 240^0 + ik \sin 240^0)$$

From the above we calculate that:

$$z_1 z_2 + z_2 z_3 + z_3 z_1 = -3 + 3k^2 \tag{1}$$

The center of the Cartesian system (also the center of the inscribed circle in $A_0 B_0 C_0$ and in the Steiner ellipse E of the triangle ABC) coincides with the barycenter of both the triangles. Hence,

$$z_1 + z_2 + z_3 = 0 \tag{2}$$

From (1) and (2) follows that

$$Q(z) = z^3 + 3(k^2 - 1)z + a_3 = 0.$$

$$Q'(z) = 3z^2 + 3(k^2 - 1) = 0.$$

The roots of Q'(z) are $\omega_1 = \sqrt{1-k^2}\ \ and\ \ \omega_2 = -\sqrt{1-k^2}$.
The circle $x^2 + y^2 = 1$ transforned by T to the ellipse

$$E : x^2 + \frac{y^2}{k^2} = 1$$

with semiaxes a=1 and b=k, therefore the foci are $\sqrt{1-k^2}\ and\ -\sqrt{1-k^2}$.

References

1. G. D. Chakerian and L. H. Lange, Geometric Extremum Problems, Mathematics Magazine, vol 44 number 2, pp 57-69.

2. Geometric Transformations, P. S. Modenov, A. S. Parkhomenko, Academic Press.